Lesson 10: Quotient Rule; Derivatives of Other Trig Functions

Lesson 10

Quotient Rule

Quotient Rule says the derivative of $h(x)=\frac{u(x)}{v(x)}$ is

$$\frac{d}{dx}[h(x)] = \frac{d}{dx}\left[\frac{u(x)}{v(x)}\right]$$
$$= \frac{u'(x)v(x) - u(x)v'(x)}{v^2(x)}$$

Ex 1: Let $h(x)=\frac{1}{x^2}$. Find $h'(x)$.

Method 1: Power Rule

Let's rewrite $h(x)$ to be
$$h(x) = x^{-2}$$
By power rule,
$$h'(x) = -2x^{-3} = \boxed{\frac{-2}{x^3}}$$

Method 2: Quotient Rule

Let $u(x) = 1 \qquad v(x) = x^2$
$\quad u'(x) = 0 \qquad v'(x) = 2x$

By Quotient Rule,
$$h'(x) = \frac{u'(x)v(x) - u(x)v'(x)}{v^2(x)}$$
$$= \frac{0\cdot(x^2) - 1(2x)}{(x^2)^2} = \frac{-2x}{x^4} = \boxed{\frac{-2}{x^3}}$$

Ex 2: Let $h(x) = \dfrac{x^2+1}{x^3-3x}$. Find $h'(x)$.

Let $u(x) \doteq x^2+1 \qquad v(x) = x^3-3x$

$\quad u'(x) = 2x \qquad V'(x) = 3x^2-3$

By Quotient Rule,

$$h'(x) = \frac{u'(x)v(x) - u(x)v'(x)}{{}^2(x)}$$

$$= \frac{2x(x^3-3x) - \boxed{(x^2+1)(3x^2-3)}}{(x^3-3x)^2}$$

Let's do the multiplication using our table trick.

	$3x^2$	-3
x^2	$3x^4$	$\cancel{-3x^2}$
1	$\cancel{3x^2}$	-3

$$h'(x) = \frac{2x(x^3-3x) - (x^2+1)(3x^2-3)}{(x^3-3x)^2}$$

$$= \frac{2x^4-6x^2-(3x^4-3)}{(x^3-3x)^2}$$

$$= \frac{2x^4-6x^2-3x^4+3}{(x^3-3x)^2}$$

$$= \boxed{\frac{-x^4-6x^2+3}{(x^3-3x)^2}}$$

Ex 3: Let $h(x) = \dfrac{\sin x}{x+\sin x}$ Find $h'(x)$.

Let $u(x) \doteq \sin x \qquad v(x) = x+\sin x$

$\quad u'(x) = \cos x \qquad V'(x) = 1+\cos x$

By Quotient Rule,

$$h'(x) = \frac{u'(x)v(x) - u(x)v'(x)}{v^2(x)}$$

$$= \frac{\cos x\,(x+\sin x) - \sin x\,(1+\cos x)}{(x+\sin x)^2}$$

$$= \frac{x\cos x + \cancel{\sin x\cos x} - \sin x - \cancel{\sin x\cos x}}{(x+\sin x)^2}$$

$$= \boxed{\frac{x\cos x - \sin x}{(x+\sin x)^2}}$$

Recall

$$\tan x = \frac{\sin x}{\cot x} \qquad \cot = \frac{\cos x}{\sin x}$$

$$\sec x = \frac{1}{\cos x} \qquad \csc x = \frac{1}{\sin x}$$

Derivatives of Other Trig Functions

$$\frac{d}{dx}[\sin x] = \cos x \qquad \frac{d}{dx}[\cos x] = -\sin x$$

$$\frac{d}{dx}[\tan x] = \sec^2 x \qquad \frac{d}{dx}[\cot x] = -\csc^2 x$$

$$\frac{d}{dx}[\sec x] = \sec x\tan x \qquad \frac{d}{dx}[\csc x] = -\csc x\cot x$$

To prove these, we use the quotient rule.

HW 10.6: Given $h(x) = 3\sin x \tan x$. Compute $h'(x)$

i.e. Find $h'(x)$ by product rule.

Let $u(x) = 3\sin x$ $\quad v(x) = \tan x$

$u'(x) = 3\cos x$ $\quad v'(x) = \sec^2 x$

By product rule,

$$h'(x) = u'(x)v(x) + u(x)v'(x)$$
$$= 3\cos x \tan x + 3\sin x \sec^2 x$$
$$= 3\cancel{\cos x}\frac{\sin x}{\cancel{\cos x}} + 3\sin x \sec^2 x$$
$$= \boxed{3\sin x + 3\sin x \sec^2 x}$$