Lesson 11: The Chain Rule Pt 1

Lesson 11: The Chain Rule

Recall Composition of Functions.

Let $y=f(g(x)) \Rightarrow$ g - inner function
f - outer function

Ex 1: Determine f and g for $y=(3x+1)^2$

$f(x)=x^2$ $g(x)=3x+1$

Check $y=f(g(x))=f(3x+1)=(3x+1)^2$ ✓

Ex 2: Determine f and g for $y=\sin^2 x=(\sin x)^2$

$f(x)=x^2$ $g(x)=\sin x$

Check $y=f(g(x))=f(\sin x)=(\sin x)^2$ ✓

Ex 3: Determine f and g for $y=\ln(2x)$

$f(x)=\ln x$ $g(x)=2x$

Check $y=f(g(x))=f(2x)=\ln(2x)$ ✓

Ex 4: Determine f and g for $y=3\sqrt{2x+1}$

$f(x)=3\sqrt{x}$ $g(x)=2x+1$

Check $y = f(g(x)) = f(2x+1) = \sqrt[3]{2x+1}$ ✓

Chain Rule

Let $y = f(\underbrace{g(x)}_{u})$

$$\frac{dy}{dx} = \frac{d}{dx}[f(u)] = \frac{df}{du} \cdot \frac{du}{dx}$$ (Format I)

$$\frac{d}{dx}[f(g(x))] = f'(g(x))\,g'(x)$$ (Format II)

Ex 1: Find y' of $y = (3x+1)^2$

Method 1: Expand and then differentiate

Recall $(a+b)^2 = a^2 + 2ab + b^2$

So $y = (3x)^2 + 2(3x)(1) + 1^2$
$= 9x^2 + 6x + 1$

$$\boxed{y' = 18x + 6}$$

Method 2: Chain Rule

Let $f(x) = x^2$ $\quad g(x) = 3x + 1$

$f'(x) = 2x$ $\quad g'(x) = 3$

By Chain Rule, $y' = f'(g(x))\,g'(x)$

$$y' = f'(3x+1)\cdot 3$$
$$= 2(3x+1)\cdot 3$$
$$= 6(3x+1)$$
$$= \boxed{18x+6}$$

Ex 2: Find y' of $y = 2\cos^3 x = 2(\cos x)^3$

Let $f(x) = 2x^3$ $\qquad g(x) = \cos x$

$f'(x) = 6x^2$ $\qquad g'(x) = -\sin x$

By Chain Rule, $y' = f'(g(x))\,g'(x)$

$$y' = f'(\cos x)\cdot(-\sin x)$$
$$= 6(\cos x)^2(-\sin x)$$
$$= \boxed{-6\cos^2 x\sin x}$$

Ex 3: Find y' of $y = \left(\frac{2x}{3x^2+x}\right)^3$

Let $f(x) = x^3$ $\qquad g(x) = \frac{2x}{3x^2+x} = \frac{2x}{x(3x+1)} = \frac{2}{3x+1}$

$f'(x) = 3x^2$

$u(x) = 2 \qquad v(x) = 3x+1$

$u'(x) = 0 \qquad v'(x) = 3$

$$g'(x) = \frac{u'(x)v(x) - u(x)v'(x)}{v^2(x)}$$

$$= \frac{0(3x+1) - 2(3)}{(3x+1)^2} = \frac{-6}{(3x+1)^2}$$

By Chain Rule, $y' = f'(g(x))\, g'(x)$

$$y' = f'\left(\frac{2}{3x+1}\right) \cdot \left(\frac{-6}{(3x+1)^2}\right)$$

$$= 3\left(\frac{2}{3x+1}\right)^2 \left(\frac{-6}{(3x+1)^2}\right)$$

$$= \frac{12}{(3x+1)^2} \cdot \frac{-6}{(3x+1)^2}$$

$$= \frac{-72}{(3x+1)^4}$$

<u>Ex4</u>: Given $y = \frac{15}{\sqrt[3]{x^2+1}}$. Find $y'(1)$.

i.e. Find y' and then plug 1.

Rewrite $y = \frac{15}{\sqrt[3]{x^2+1}} = \frac{15}{(x^2+1)^{1/3}} = 15(x^2+1)^{-1/3}$

Let $f(x) = 15x^{-1/3}$ $\qquad g(x) = x^2+1$

$f'(x) = 15\left(-\frac{1}{3}\right)x^{-4/3}$ $\qquad g'(x) = 2x$

$= -5x^{-4/3}$

By Chain Rule, $y' = f'(g(x))\, g'(x)$

$$y' = f'(x^2+1)(2x)$$

$$= -5(x^2+1)^{-4/3}(2x)$$

$$= \frac{-10x}{(x^2+1)^{4/3}}$$ Plug $x=1$!

$$y'(1) = \frac{-10}{(1^2+1)^{4/3}} = \frac{-10}{(2)^{4/3}} = \frac{-5}{1} \cdot \frac{2}{2^{4/3}} = \frac{-5}{2^{1/3}} = \frac{-5}{\sqrt[3]{2}}$$