Lesson 12: The Chain Rule Pt 2; Derivative of Natural Logarithmic Function

Lesson 12: More Chain Rule

Recall when $y=f(g(x))$ then

$$y'=f'(g(x))g'(x)$$

Trig & Exponential Functions w/ Chain Rule

Note $\sin^2 x = (\sin x)^2$ ← Check by plugging $x=\frac{\pi}{2}$ into both.

$\neq$

$\sin(x^2)$ ← Are they equal?

Ex 1: Find y' when $y=\sin(x^2)$

Let $f(x)=\sin x$ $\quad g(x)=x^2$

$f'(x)=\cos x$ $\quad g'(x)=2x$

By Chain Rule,

$$y'=f'(g(x))g'(x)$$

$$=f'(x^2)(2x)$$

$$=2x\cos(x^2)$$

Ex2: Find y' when $y = \sec(-2x+1)\tan(3x)$.

Note this is a product $\Rightarrow$ Product Rule

Let $u(x) = \sec(-2x+1)$ $\quad v(x) = \tan(3x)$

Note $u(x)$ & $v(x)$ are both compositions so we have to use Chain Rule for both $u(x)$ & $v(x)$.

Let's first do $u(x)$, and then $v(x)$.

Let $u(x) = \sec(-2x+1)$

$$f(x) = \sec x \qquad g(x) = -2x+1$$
$$f'(x) = \sec x \tan x \qquad g'(x) = -2$$

$$\begin{aligned} u'(x) &= f'(g(x))\,g'(x) \\ &= f'(-2x+1)\cdot(-2) \\ &= -2\sec(-2x+1)\tan(-2x+1) \end{aligned}$$

Let $v(x) = \tan(3x)$

$$f(x) = \tan x \qquad g(x) = 3x$$
$$f'(x) = \sec^2 x \qquad g'(x) = 3$$

$$\begin{aligned} v'(x) &= f'(g(x))\,g'(x) \\ &= f'(3x)\cdot 3 \\ &= 3\sec^2(3x) \end{aligned}$$

By Product Rule,

$$\begin{aligned} y' &= u'(x)v(x) + u(x)v'(x) \\ &= -2\sec(-2x+1)\tan(-2x+1)\tan(3x) \\ &\quad + 3\sec^2(3x)\sec(-2x+1) \end{aligned}$$

Ex 3: Find y' when $y = e^{(1-2x)^4} = \exp[(1-2x)^4]$

Note this a chain rule within a chain rule.

Let $f(x) = e^x$ $\quad g(x) = (1-2x)^4$

$f'(x) = e^x$

$a(x) = x^4 \qquad b(x) = 1-2x$

$a'(x) = 4x^3 \qquad b'(x) = -2$

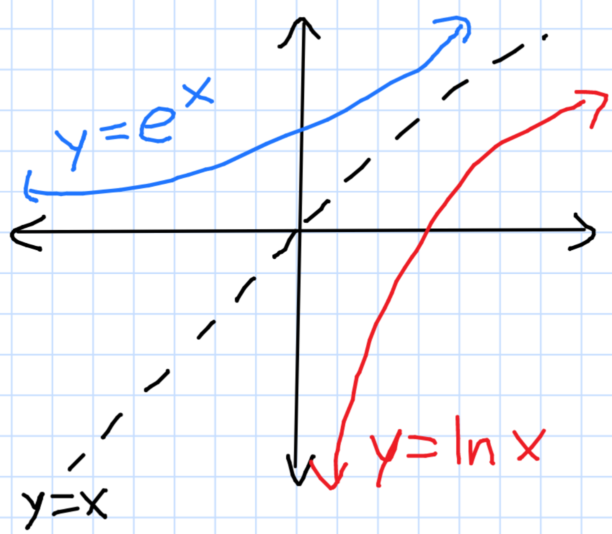
$$\begin{aligned} g'(x) &= a'(b(x))\, b'(x) \\ &= a'(1-2x)(-2) \\ &= 4(1-2x)^3(-2) \\ &= -8(1-2x)^3 \end{aligned}$$

By Chain Rule,

$$\begin{aligned} y &= f'(g(x))\, g'(x) \\ &= f'((1-2x)^4) \cdot (-8)(1-2x)^3 \\ &= -8\exp[(1-2x)^4](1-2x)^3 \end{aligned}$$

Derivative of Logarithmic Function

Recall e^x and $\ln x$ are inverses.



Furthermore,

$$\text{Slope of } \ln x = \frac{1}{\text{Slope of } e^x}$$

So for $x > 0$,

$$\frac{d}{dx}[\ln x] = \frac{1}{x}$$

Ex 1: Find y' when $y = x\ln x$

Note this is a product rule.

Let $u(x) = x$ $\quad v(x) = \ln x$

$u'(x) = 1$ $\quad v'(x) = \frac{1}{x}$

By Product Rule,

$$y' = u'(x)v(x) + u(x)v'(x)$$
$$= 1 \cdot \ln x + \cancel{x} \cdot \frac{1}{\cancel{x}}$$
$$= \ln x + 1$$

Ex 2: Find y' when $y = \ln(3x^2 + x + 1)$

Note this is a chain rule.

Let $f(x) = \ln x$ $\quad g(x) = 3x^2 + x + 1$

$f'(x) = \frac{1}{x}$ $\quad g'(x) = 6x + 1$

By Chain Rule,

$$y' = f'(g(x))g'(x)$$
$$= f'(3x^2 + x + 1)(6x + 1)$$
$$= \frac{1}{3x^2 + x + 1} \cdot (6x + 1) = \frac{6x + 1}{3x^2 + x + 1}$$

Recall

$$\left.\begin{aligned} &\ln\left(\frac{x}{y}\right) = \ln x - \ln y \\ &\ln(xy) = \ln x + \ln y \\ &\ln(x^m) = m\ln x \\ &\ln(1) = 0 \\ &\ln(e^x) = x \end{aligned}\right\} \text{Memorize}$$

Ex 3: Find y' when $y = \ln \sqrt[3]{\frac{x^2+1}{2x-1}}$

Method 1: Chain Rule with

$F(x) = \ln x \qquad \underbrace{g(x) = \left(\frac{x^2+1}{2x-1}\right)^{1/3}}_{\text{Chain Rule Again}}$

$a(x) = x^{1/3} \qquad \underbrace{b(x) = \frac{x^2+1}{2x-1}}_{\text{Apply Quotient Rule}}$

or what if we do the following

Method 2: Rewrite y w/ Logarithmic Rules

$$y = \ln \sqrt[3]{\frac{x^2+1}{2x-1}} = \ln\left(\frac{x^2+1}{2x-1}\right)^{1/3} = \frac{1}{3}\ln\left(\frac{x^2+1}{2x-1}\right)$$

$$= \frac{1}{3}\left[\ln(x^2+1) - \ln(2x-1)\right]$$

$$= \frac{1}{3}\ln(x^2+1) - \frac{1}{3}\ln(2x-1)$$

$$y' = \frac{1}{3} \cdot \frac{1}{x^2+1} \cdot 2x - \frac{1}{3} \cdot \frac{1}{2x-1} \cdot 2$$

$$= \frac{2x}{3(x^2+1)} - \frac{2}{3(2x-1)}$$