# Lesson 14: Implicit Differentiation

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Explicit Form: $y = f(x)$

Implicit Form: when a function is NOT written in explicit form

ex. ① $y - 2x = 1$ ② $x^2 + y^2 = 2$ ③ $y^3 + y - 1 = x$

To differentiate functions of this kind, we use a technique called implicit differentiation.

We namely use this technique when solving for y is particularly messy.

Ex 1: Use implicit differentiation to find slope of tangent line of

$$x^2 - y^2 = 4x + 8y \text{ @ } (0,0) \quad \left[\text{i.e. } \frac{dy}{dx}(0,0)\right]$$

$$\frac{d}{dx}(x^2 - y^2) = \frac{d}{dx}(4x + 8y)$$

$$\frac{d}{dx}(x^2) - \frac{d}{dx}(y^2) = \frac{d}{dx}(4x) + \frac{d}{dx}(8y)$$

$$2x \cancel{\frac{dx}{dx}}^{1} - 2y\frac{dy}{dx} = 4\cancel{\frac{dx}{dx}}^{1} + 8\frac{dy}{dx}$$

Remember $\frac{dx}{dx} = 1$

$$2x - 2y\frac{dy}{dx} = 4 + 8\frac{dy}{dx}$$

Now we just need to solve for dy/dx

$$\begin{array}{l} 2x - \cancel{2y\frac{dy}{dx}} = \cancel{4} + 8\frac{dy}{dx} \\ -4 \quad \cancel{+2y\frac{dy}{dx}} \quad \cancel{-4} \quad +2y\frac{dy}{dx} \\ \hline 2x - 4 = 8\frac{dy}{dx} + 2y\frac{dy}{dx} \end{array}$$

$$\frac{2x-4}{8+2y} = \frac{\cancel{(8+2y)}\frac{dy}{dx}}{\cancel{8+2y}} \Rightarrow \frac{dy}{dx} = \frac{2x-4}{8+2y}$$

$$\frac{dy}{dx}(0,0) = \frac{2(0)-4}{8+2(0)} = \frac{-4}{8} = \boxed{-\frac{1}{2}}$$

<u>Ex2</u>: Use implicit differentiation to find $dy/dx$ of

$$yx^2 + e^y = x$$

$$\frac{d}{dx}(yx^2 + e^y) = \frac{d}{dx}(x)$$

$$\frac{d}{dx}(yx^2) + \frac{d}{dx}(e^y) = \frac{d}{dx}(x)$$

Is this a function of x or y? Both

Is it a product or quotient? Product $\Rightarrow$ Use product rule

$$\frac{d}{dx}(y) \cdot x^2 + y \cdot \frac{d}{dx}(x^2) + \frac{d}{dx}(e^y) = \frac{d}{dx}(x)$$

$$1 \cdot \frac{dy}{dx} \cdot x^2 + y \cdot 2x \cdot \cancel{\frac{dx}{dx}}^{1} + e^y\frac{dy}{dx} = 1 \cdot \cancel{\frac{dx}{dx}}^{1}$$

$$\begin{array}{l} x^2\frac{dy}{dx} + \cancel{2xy} + e^y\frac{dy}{dx} = 1 \\ \quad \cancel{-2xy} \qquad\qquad -2xy \\ \hline x^2\frac{dy}{dx} + e^y\frac{dy}{dx} = 1 - 2xy \end{array}$$

$$\frac{\cancel{(x^2+e^y)}\frac{dy}{dx}}{\cancel{x^2+e^y}} = \frac{1-2xy}{x^2+e^y}$$

$$\frac{dy}{dx} = \frac{1-2xy}{x^2+e^y}$$

## Extra Credit 2

$$\frac{d}{dx}(xy) = \frac{d}{dx}(x)\cdot y + x\cdot\frac{d}{dx}(y)$$

$$= 1\cdot\cancel{\frac{dx}{dx}}^{\,1}\cdot y + x\cdot 1\cdot\frac{dy}{dx}$$

$$= y + x\frac{dy}{dx}$$

Basically product rule

Hint for $\frac{d}{dx}\left(\frac{x}{y}\right)$: Use quotient rule.

<u>Ex3</u>: Use implicit differentiation to find $dy/dx$ of

$$4\sin x\cos y = 3$$

$$\frac{d}{dx}(4\sin x\cos y) = \frac{d}{dx}(3)$$

$$\frac{d}{dx}(4\sin x)\cdot\cos y + 4\sin x\cdot\frac{d}{dx}(\cos y) = \frac{d}{dx}(3)$$

$$4\cos x\,\cancel{\frac{dx}{dx}}^{\,1}\cdot\cos y + 4\sin x\cdot(-\sin y)\frac{dy}{dx} = 0$$

$$4\cos x\cos y - 4\sin x\sin y\,\frac{dy}{dx} = 0$$

$$4\cos x\cos y = 4\sin x\sin y\,\frac{dy}{dx}$$

$$\frac{\cancel{4}\cos x\cos y}{\cancel{4}\sin x\sin y} = \frac{\cancel{4}\cancel{\sin x}\,\cancel{\sin y}\,\frac{dy}{dx}}{\cancel{4}\cancel{\sin x}\,\cancel{\sin y}}$$

$$\frac{dy}{dx} = \frac{\cos x\cos y}{\sin x\sin y} = \frac{\cos x}{\sin x}\cdot\frac{\cos y}{\sin y} = \boxed{\cot x\cdot\cot y}$$

<u>Ex 4</u>: Use implicit differentiation to find dy/dx of

$$6\tan(2x+3y) = 11x$$

$$\underbrace{\frac{d}{dx}\left(6\tan(2x+3y)\right)}_{\text{Use chain rule}} = \frac{d}{dx}(11x)$$

$$6\sec^2(2x+3y)\cdot\frac{d}{dx}(2x+3y) = \frac{d}{dx}(11x)$$

$$6\sec^2(2x+3y)\left[2\,\cancel{\frac{dx}{dx}}^{\,1} + 3\frac{dy}{dx}\right] = 11\,\cancel{\frac{dx}{dx}}^{\,1}$$

$$6\sec^2(2x+3y)\left[2 + 3\frac{dy}{dx}\right] = 11$$

$$\frac{6\sec^2(2x+3y)\left[2 + 3\frac{dy}{dx}\right]}{6\sec^2(2x+3y)} = \frac{11}{6\sec^2(2x+3y)}$$

$$\cancel{2} + 3\frac{dy}{dx} = \frac{11}{6}\cos^2(2x+3y)$$

$$\cancel{-2} \qquad\qquad -2$$

$$3\frac{dy}{dx} = \frac{11}{6}\cos^2(2x+3y) - 2$$

$$\frac{1}{\cancel{3}}\left(\cancel{3}\frac{dy}{dx}\right)=\frac{1}{3}\left(\frac{11}{6}\cos^2(2x+3y)-2\right)$$

$$\frac{dy}{dx}=\boxed{\frac{11}{18}\cos^2(2x+3y)-\frac{2}{3}}$$

## Formal Proof of Why $\frac{d}{dx}[\ln x]=\frac{1}{x}$

Let $y=\ln x$. Note $y=\ln x \iff e^y=x$.

Differentiate.

$$\frac{d}{dx}(e^y)=\frac{d}{dx}(x)$$

$$e^y\frac{dy}{dx}=1\cdot\frac{dx}{dx}$$

$$e^y\frac{dy}{dx}=1$$

But recall ↗. So

$$\frac{dy}{dx}=\frac{1}{x}.$$

Hence

$$\frac{d}{dx}[\ln x]=\frac{1}{x}$$