MA 16010 LESSONS 15-16: RELATED RATES

This space is left for you to take your own notes.

Related Rates are word problems that use implicit differentiation.

We will be taking the derivative of equations with respect to time, t.

Recipe for Solving a Related Rates Problem

Step 1: Draw a good picture. Label all constant values and give variable names to any changing quantities.

Step 2: Determine what information you **KNOW** and what you **WANT** to find.

Step 3: Find an equation relating the relevant variables. This usually involves a formula from geometry, similar triangles, the Pythagorean Theorem, or a formula from trigonometry. **Use your picture!**

Step 4: Use implicit differentiation to differentiate the equation with respect to time *t*.

Step 5: Substitute in what you **KNOW** from **Step 2** and any information that your equation in **Step 3** can give you and solve for the quantity you **WANT**. Do **NOT** substitute before this step!

Some Useful Formulas

Right Triangle *Pythagorean Theorem:* $a^2 + b^2 = c^2$	Triangle $A = \frac{1}{2}bh$ $P = a + b + c$	Trapezoid $A = \frac{1}{2}(a+b)h$
Rectangular Box $V = lwh$ $S = 2(hl + lw + hw)$	Rectangle $A = lw$ $P = 2l + 2w$	Circle $A = \pi r^2$ $C = 2\pi r$
Right Circular Cylinder $V = \pi r^2 h$ $SA = 2\pi rh$	Sphere $V = \frac{4}{3}\pi r^3$ $S = 4\pi r^2$	Cone $V = \frac{1}{3}\pi r^2 h$ $SA = \pi rl + \pi r^2$

Example 1: If x and y are both functions of t and $x + y^3 = 2$.

(a) Find $\frac{dy}{dt}$ when $\frac{dx}{dt}$ = -2 and $y = 1$.

$$\frac{d}{dt}(x+y^3) = \frac{d}{dt}(2)$$

$$\frac{d}{dt}(x) + \frac{d}{dt}(y^3) = \frac{d}{dt}(2)$$

$$1 \cdot \frac{dx}{dt} + 3y^2\frac{dy}{dt} = 0$$

$$-2 + 3 \cdot 1^2 \frac{dy}{dt} = 0$$

$$-2 + 3\frac{dy}{dt} = 0$$

$$3\frac{dy}{dt} = 2$$

$$\frac{dy}{dt} = \frac{2}{3}$$

(b) Find $\frac{dx}{dt}$ when $\frac{dy}{dt}$ = 3 and $x = 1$

$$\frac{dx}{dt} + 3y^2\frac{dy}{dt} = 0$$

$$\frac{dx}{dt} + 3y^2 \cdot 3 = 0$$

$$\frac{dx}{dt} + 9y^2 = 0$$

$$\frac{dx}{dt} = -9y^2$$

$$\frac{dx}{dt} = -9 \cdot 1^2 = -9$$

$$x + y^3 = 2$$

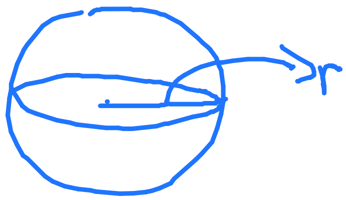
$$1 + y^3 = 2$$

$$y^3 = 1$$

$$y = 1$$

Example 2: A spherical balloon is being deflated at a constant rate of 20 cubic cm per second. How fast is the radius of the balloon changing at the instant when the balloon's radius is 12 cm?

Step 1: Draw a picture. Label all constant values and give variable names to any changing quantities.



Step 2: Determine what information you **KNOW** and what you WANT to find.

KNOW: $\frac{dV}{dt} = -20 \ \frac{cm^3}{s}$ **WANT:** $\left.\frac{dr}{dt}\right|_{r=12}$

Step 3: Find an equation relating the relevant variables.

$$V = \frac{4}{3}\pi r^3$$

Step 4: Use implicit differentiation to differentiate the equation with respect to time *t*.

$$\frac{d}{dt}(V) = \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right)$$

$$\frac{dV}{dt} = \frac{4}{\cancel{3}}\pi(\cancel{3}r^2)\frac{dr}{dt}$$

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

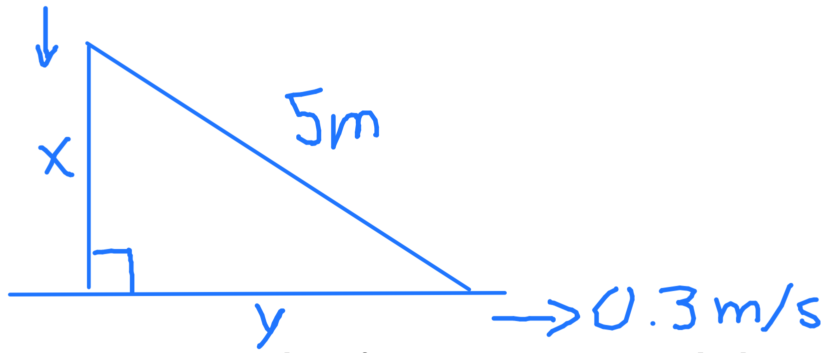
Step 5: Substitute in what you **KNOW** from **Step 2** and any information that your equation in **Step 3** can give you and solve for the quantity you WANT**.**

$$\frac{-20}{4\pi(12)^2} = \frac{4\pi(12)^2 \frac{dr}{dt}}{4\pi(12)^2}$$

$$\frac{-5}{(12)^2\pi} = \boxed{\frac{dr}{dt} = \frac{-5}{121\pi}}$$

<u>Example 3:</u> A ladder 5 meters long rests on horizontal ground and leans against a vertical wall. The foot of the ladder is pulled away from the wall at the rate of 0.3 m/sec. How fast is the top sliding down the wall when the foot of the ladder is 3 m from the wall?

Step 1: Draw a picture. Label all constant values and give variable names to any changing quantities.



Step 2: Determine what information you **KNOW** and what you WANT to find.

KNOW: $\frac{dy}{dt} = 0.3 \ \frac{m}{s}$ **WANT:** $\left.\frac{dx}{dt}\right|_{y=3}$

Step 3: Find an equation relating the relevant variables.

$$x^2 + y^2 = 5^2 = 25$$

Step 4: Use implicit differentiation to differentiate the equation with respect to time *t*.

$$\frac{d}{dt}(x^2 + y^2) = \frac{d}{dt}(25)$$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$2x\frac{dx}{dt} = -2y\frac{dy}{dt}$$

$$\frac{\cancel{2}x}{\cancel{2}x}\frac{dx}{dt} = -\frac{\cancel{2}y}{\cancel{2}x}\frac{dy}{dt}$$

$$\frac{dx}{dt} = -\frac{y}{x}\frac{dy}{dt}$$

Step 5: Substitute in what you **KNOW** from **Step 2** and any information that your equation in **Step 3** can give you and solve for the quantity you WANT.

$$\frac{dx}{dt} = -\frac{3}{x} \cdot 0.3 = -\frac{0.9}{x}$$

$$\frac{dx}{dt} = -\frac{0.9}{4} = \boxed{-0.225}$$

Find x by plugging y=3 into

$$x^2 + y^2 = 25$$

$$x^2 + 3^2 = 25$$

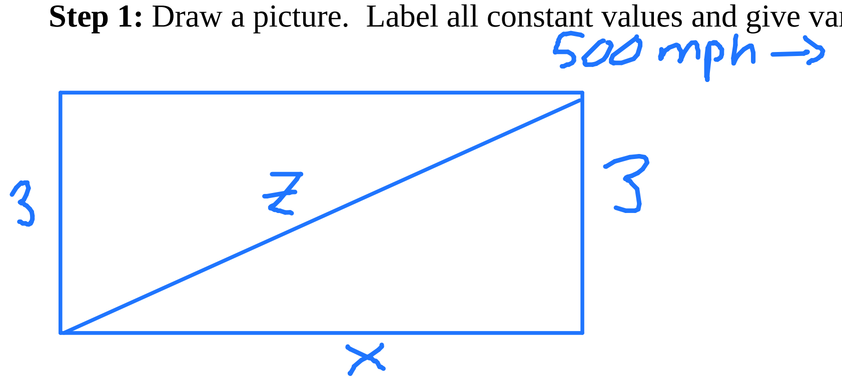
$$x^2 = 25 - 9 = 16$$

$$x = 4$$

Example 4: A plane is flying directly away from you at 500 mph at an altitude of 3 miles.

(1) How fast is the plane's distance from you increasing at the moment when the plane is flying over a point on the ground 4 miles from you?

Step 1: Draw a picture. Label all constant values and give variable names to any changing quantities.



Step 2: Determine what information you **KNOW** and what you WANT to find.

KNOW: $\frac{dx}{dt} = 500 \text{ mph}$ **WANT:** $\left.\frac{dz}{dt}\right|_{x=4}$

Step 3: Find an equation relating the relevant variables.

$$x^2 + 3^2 = z^2 \iff x^2 + 9 = z^2$$

Step 4: Use implicit differentiation to differentiate the equation with respect to time *t*.

$$2x\frac{dx}{dt} = 2z\frac{dz}{dt}$$

$$\frac{\cancel{2}x}{\cancel{2}z}\frac{dx}{dt} = \frac{dz}{dt}$$

$$\frac{dz}{dt} = \frac{x}{z}\frac{dx}{dt}$$

Step 5: Substitute in what you **KNOW** from **Step 2** and any information that your equation in **Step 3** can give you and solve for the quantity you WANT.

$$\frac{dz}{dt} = \frac{4}{z}\cdot(500) = \frac{2000}{z}$$

$$= \frac{2000}{5}$$

$$= \boxed{400}$$

Find z by plugging $x = 4$ into

$$x^2 + 9 = z^2$$

$$4^2 + 9 = z^2$$

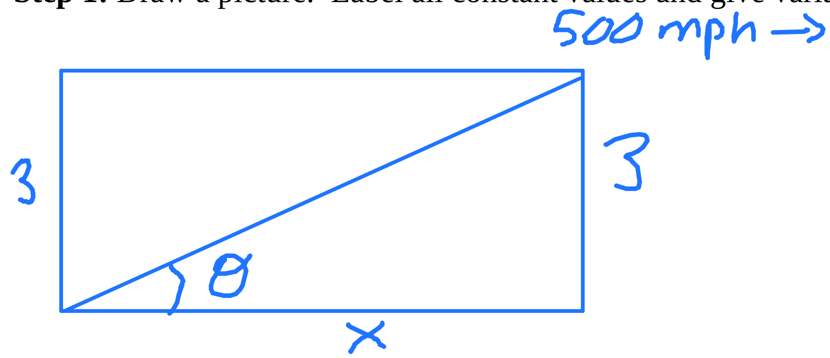
$$25 = z^2$$

$$5 = z$$

Example 4: A plane is flying directly away from you at 500 mph at an altitude of 3 miles.

(2) How fast is the angle of elevation changing when it is $\pi/3$?

Step 1: Draw a picture. Label all constant values and give variable names to any changing quantities.



Step 2: Determine what information you **KNOW** and what you WANT to find.

KNOW: $\frac{dx}{dt} = 500 \text{ mph}$ **WANT:** $\left.\frac{d\theta}{dt}\right|_{\theta = \pi/3}$

Step 3: Find an equation relating the relevant variables.

$$\tan\theta = \frac{3}{x} = 3x^{-1}$$

Step 4: Use implicit differentiation to differentiate the equation with respect to time *t*.

$$\sec^2\theta \frac{d\theta}{dt} = -3x^{-2}\frac{dx}{dt}$$

$$\sec^2\theta \frac{d\theta}{dt} = -\frac{3}{x^2}\frac{dx}{dt}$$

$$\frac{d\theta}{dt} = -\frac{3}{x^2}\cos^2\theta \frac{dx}{dt}$$

Step 5: Substitute in what you **KNOW** from **Step 2** and any information that your equation in **Step 3** can give you and solve for the quantity you WANT.

$$\frac{d\theta}{dt} = \frac{-3}{x^2}\left(\cos\left(\frac{\pi}{3}\right)\right)^2(500)$$

$$= -\frac{3}{x^2}\left(\frac{1}{2}\right)^2(500)$$

$$= -\frac{375}{x^2}$$

$$= -\frac{375}{3} = \boxed{-125}$$

Find x by plugging $\theta = \pi/3$ into

$$\tan\theta = 3/x$$

$$\tan\left(\frac{\pi}{3}\right) = \frac{3}{x}$$

$$\frac{\sqrt{3}}{1} = \frac{3}{x}$$

$$\sqrt{3}\,x = 3$$

$$x = \frac{3}{\sqrt{3}} \Rightarrow x^2 = \frac{9}{3} = 3$$

<u>HW 15.5:</u> A cylindrical tank standing upright (with one circular base on the ground) has a radius of 22 cm for the base. How fast does the water level in the tank drop when the water is being drained at 28 cm^3/*sec*? Note: The formula right circular cylinder is V = $\pi r^2 h$.

Step 1: Draw a picture. Label all constant values and give variable names to any changing quantities.

Step 2: Determine what information you **KNOW** and what you WANT to find.

KNOW: $\frac{dV}{dt} = -28 \ \frac{cm^3}{s}$ **WANT:** $\frac{dh}{dt}$

Step 3: Find an equation relating the relevant variables.

$$V = \pi r^2 h \iff V = \pi (22)^2 h$$

Step 4: Use implicit differentiation to differentiate the equation with respect to time *t*.

$$\frac{dV}{dt} = (22)^2 \pi \frac{dh}{dt}$$

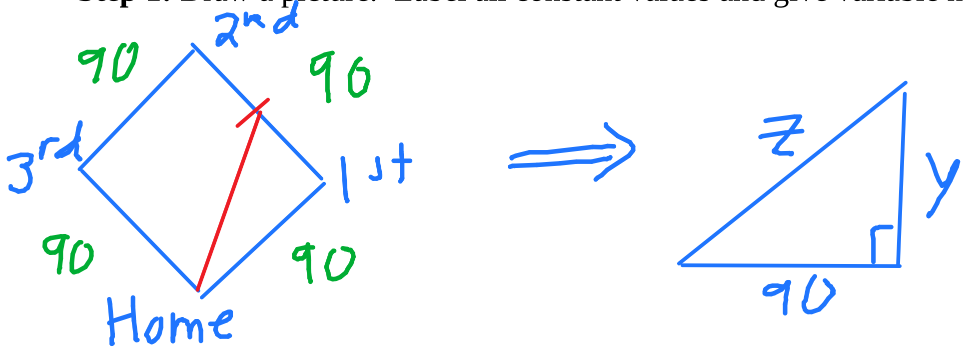
$$\frac{dh}{dt} = \frac{1}{(22)^2 \pi} \frac{dV}{dt}$$

Step 5: Substitute in what you **KNOW** from **Step 2** and any information that your equation in **Step 3** can give you and solve for the quantity you WANT**.**

$$\frac{dh}{dt} = \frac{1}{(22)^2 \pi}(-28) = \frac{-\cancel{4} \cdot 7}{\cancel{2^2} \cdot 11^2 \pi} = \boxed{\frac{-7}{121\pi}}$$

HW 16.3: A baseball diamond is a square 90 ft on a side. A player runs from first base to second base at 14 ft/sec. At what rate is the player's distance from home base increasing when he is halfway from first to second base?

Step 1: Draw a picture. Label all constant values and give variable names to any changing quantities.



Step 2: Determine what information you **KNOW** and what you WANT to find.

KNOW: $\frac{dy}{dt} = 14\ \frac{ft}{s}$

WANT: $\left.\frac{dz}{dt}\right|_{y=\frac{90}{2}=45\text{ ft}}$

Step 3: Find an equation relating the relevant variables.

$$90^2 + y^2 = z^2$$

Step 4: Use implicit differentiation to differentiate the equation with respect to time *t*.

$$\cancel{2}y\frac{dy}{dt} = \cancel{2}z\frac{dz}{dt}$$

$$\frac{y}{z}\frac{dy}{dt} = \frac{dz}{dt}$$

Step 5: Substitute in what you **KNOW** from **Step 2** and any information that your equation in **Step 3** can give you and solve for the quantity you WANT.

$$\frac{dz}{dt} = \frac{45}{z}(14)$$

$$= \frac{\cancel{45}\cdot 14}{\cancel{45}\sqrt{5}} = \boxed{\frac{14}{\sqrt{5}}}$$

Find z by plugging $y=45$ into $90^2 + y^2 = z^2$

$$2^2 \cdot 45^2 + 45^2 = z^2$$

$$45^2(2^2+1) = z^2$$

$$45\sqrt{5} = z$$