Lesson 19: Concavity, Inflection Points & the Second Derivative Test

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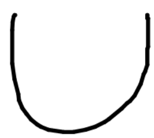
Concave Up



Like a cup

Concave Down

Like a frown

Concavity of a Function:

Suppose $f''(x)$ exists on an open interval, I. Then

ⓐ If $\boxed{f''(x) > 0}$ for all x in I, then $f(x)$ is $\boxed{\text{concave up}}$ on I.

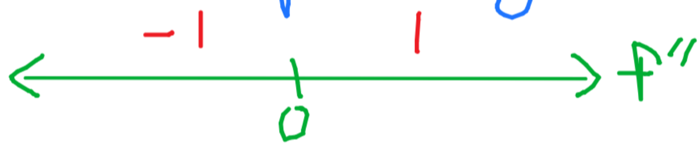
ⓑ If $\boxed{f''(x) < 0}$ for all x in I, then $f(x)$ is $\boxed{\text{concave down}}$ on I.

Ex 1: Determine the largest open interval(s) on which $f(x) = x^3 - x$ is concave up or down.

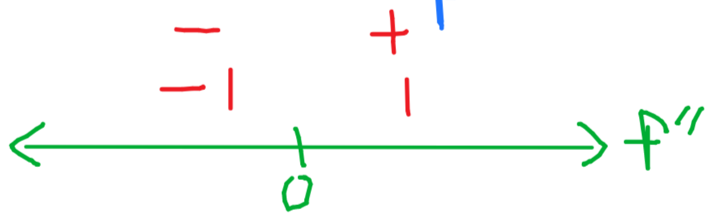
Step 1: Find when $f''(x) = 0$

$$f'(x) = 3x^2 - 1$$
$$f''(x) = 6x = 0$$
$$x = 0$$

Step 2+3: Draw a # line with the pts from ① Determine test pts using ②

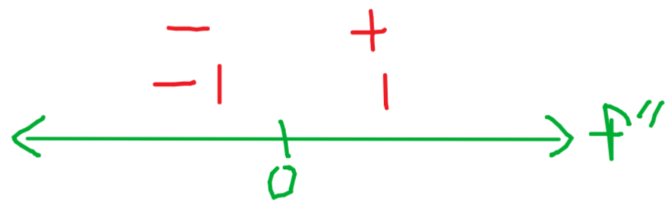


Step 4: Plug those values into f'' to determine whether it's positive or negative



$$f''(x) = 6x$$
$$f''(-1) = 6(-1)$$
−
$$f''(1) = 6(1)$$
+

Step 5: Use definition of concativity



Concave Up: $(0, \infty)$

Concave Down: $(-\infty, 0)$

Ex 2: Let $f(x) = \frac{1}{12}x^4 - \frac{1}{3}x^3$.

ⓐ Determine the largest open interval(s) on which $f(x)$ is increasing or decreasing.

Step 1: Find when $f'(x) = 0$

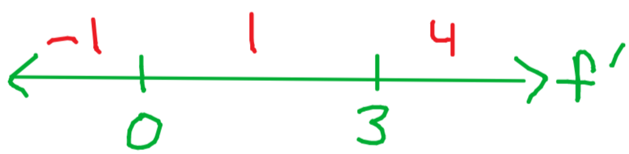
$$f'(x) = \frac{4}{12}x^3 - \frac{3}{3}x^2 = \frac{1}{3}x^3 - x^2 = 0$$

$$x^2\left(\frac{1}{3}x - 1\right) = 0$$

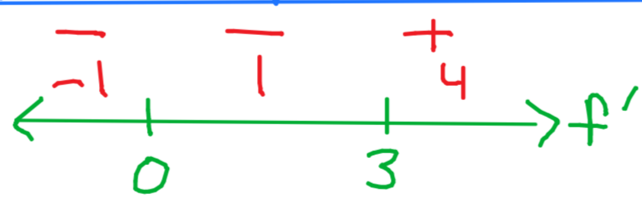
$x^2 = 0$	$\frac{1}{3}x - 1 = 0$
$x = 0$	$\frac{1}{3}x = 1$
	$x = 3$

Step 2+3: Draw a # line with the pts from ①

Determine test pts using ②

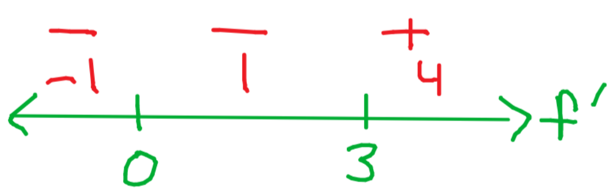


Step 4: Plug those values into f' to determine whether it's positive or negative



$f'(x) = x^2\left(\frac{1}{3}x - 1\right)$

Step 5: Use definition of increasing/decreasing



Increasing: $(3, \infty)$

Decreasing: $(-\infty, 3)$

Ex 2: Let $f(x) = \frac{1}{12}x^4 - \frac{1}{3}x^3$.

(b) Determine the largest open interval(s) on which $f(x)$ is concave up or down.

Step 1: Find when $f''(x) = 0$

$$f'(x) = \frac{4}{12}x^3 - \frac{3}{3}x^2 = \frac{1}{3}x^3 - x^2$$

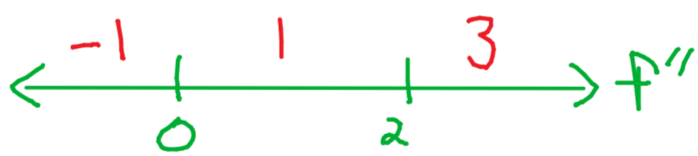
$$f''(x) = \frac{3}{3}x^2 - 2x = x^2 - 2x = 0$$

$$x(x-2) = 0$$

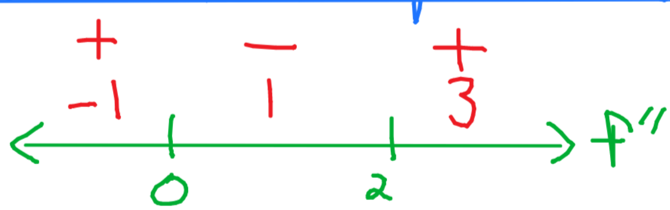
$$x = 0 \quad \Big| \quad x - 2 = 0$$
$$x = 2$$

Step 2: Draw a # w/ the pts from (1)

Step 3: Determine test pts using (2).



Step 4: Plug those values into f'' to determine whether it's positive or negative



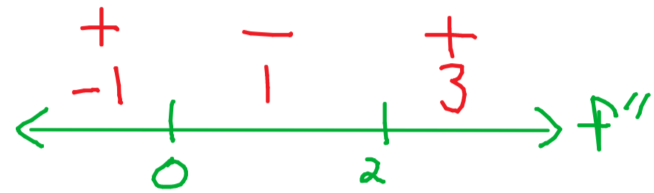
$$f''(x) = x(x-2)$$

$$f''(-1) = -1(-1-2)$$
$$- \cdot - = +$$

$$f''(1) = 1(1-2)$$
$$+ \cdot - = -$$

$$f''(3) = 3(3-2)$$
$$+ \cdot + = +$$

Step 5: Use definition of concavity



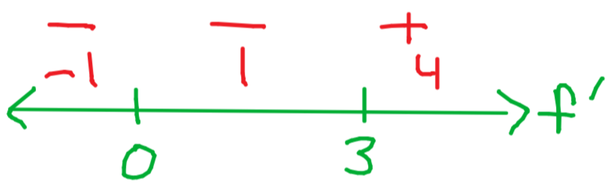
Concave Up: $(-\infty, 0) \cup (2, \infty)$

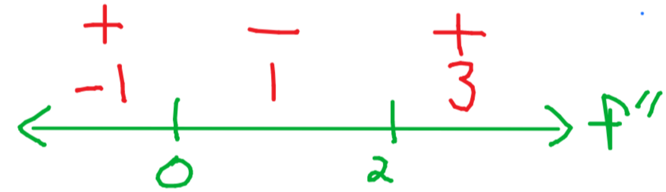
Concave Down: $(0, 2)$

Ex 2: Let $f(x) = \frac{1}{12}x^4 - \frac{1}{3}x^3$.

ⓒ Determine the largest open interval(s) on which $f(x)$ is decreasing and concave up.

Use your # lines from ⓐ and ⓑ





	$(-\infty, 0)$	$(0, 2)$	$(2, 3)$	$(3, \infty)$
f'	−	−	−	+
f''	+	−	+	+

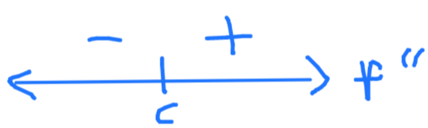
Answer: $(-\infty, 0) \cup (2, 3)$

Steps in Finding the Inflection Pts

① Find the pts on the curve where the second derivative is 0 or DNE.

i.e. Find where $f''(x) = 0$ and $f''(x)$ DNE

② Test whether the concavity changes @ these pts

i.e. 

or 

where c is a pt found in ①

Ex 3: Find the inflection pt(s) of $f(x)=\frac{3}{5}x^5-x^4$ if they exist.

Step 1: Find when $f''(x)=0$

$$f'(x)=\frac{3}{5}(5)x^4-4x^3=3x^4-4x^3$$

$$f''(x)=12x^3-12x^2=0$$

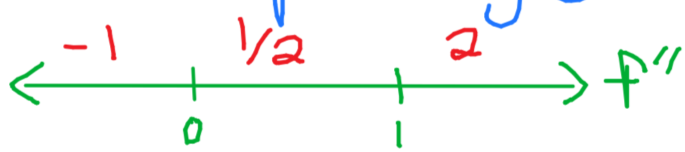
$$12x^2(x-1)=0$$

$$12x^2=0 \quad \Big| \quad x-1=0$$

$$x=0 \quad \Big| \quad x=1$$

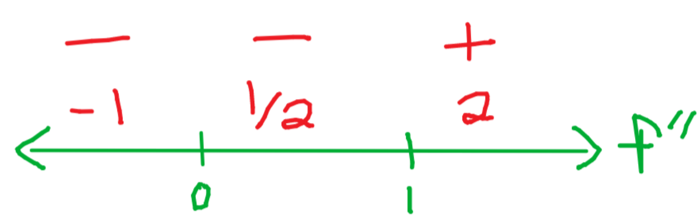
Step 2+3: Draw a # line with the pts from ①

Determine test pts using ②



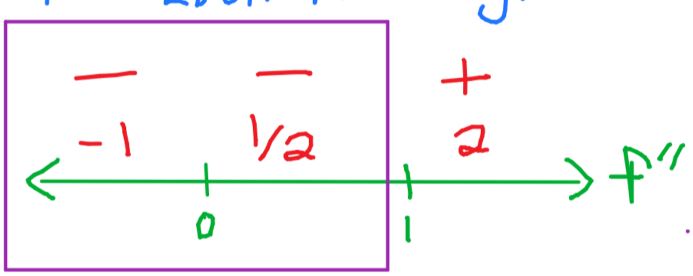
Step 4 Plug those values into f' to determine whether it's positive or negative

$$f''(x)=12x^2(x-1)$$



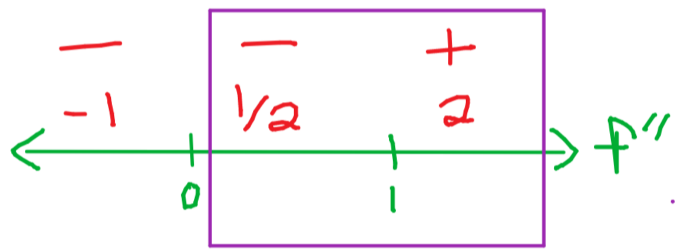
Step 5: Use definition of inflection pts.

i.e. Look for sign change.



Does the sign change?

No ⇒ x=0 NOT Inflection Pt



Does the sign change?
Yes ⇒ $x = 1$ __IS__ Inflection pt

Second Derivative Test

Let $f(x)$ be a function such that $f'(c) = 0$ and $f''(x)$ exists on an open interval containing c.

① If $f''(c) > 0$ then $f(x)$ has a relative min $f(c)$ at $x = c$.

② If $f''(c) < 0$ then $f(x)$ has a relative max $f(c)$ at $x = c$.

If $f''(c) = 0$, the Second Derivative Test does not apply or fail.

It does not necessarily mean that we have neither a relative max or min at these pts.

It just means you need to apply First Derivative Test.

Ex 4: Find all the relative extrema of $f(x) = x^3 - x$ if they exist.

Step 1: Find $f'(x) = 0$

$$f'(x) = 3x^2 - 1 = 0$$
$$x^2 = 1/3$$
$$x = \pm\sqrt{1/3}$$

Step 2: Find $f''(x)$

$$f''(x) = 6x$$

Step 3: Plug ① into ②

$$f''\left(\sqrt{\tfrac{1}{3}}\right) = 6\sqrt{\tfrac{1}{3}}$$
$$f''\left(-\sqrt{\tfrac{1}{3}}\right) = 6\left(-\sqrt{\tfrac{1}{3}}\right)$$

Step 4: Use Second Derivative Test

$$f''\left(\sqrt{\tfrac{1}{3}}\right) = 6\sqrt{\tfrac{1}{3}} > 0 \quad \cup$$
$$\Rightarrow \text{rel min}$$

$$f''\left(-\sqrt{\tfrac{1}{3}}\right) = 6\left(-\sqrt{\tfrac{1}{3}}\right) < 0 \quad \cap$$
$$\Rightarrow \text{rel max}$$

Ex 5: Find all the relative extrema of $f(x) = \frac{3}{5}x^5 - x^4$ if they exist.

Step 1: Find $f'(x) = 0$

$$f'(x) = \frac{3}{5}(5)x^4 - 4x^3$$
$$3x^4 - 4x^3 = 0$$
$$x^3(3x-4) = 0$$

$x^3 = 0$, $x = 0$ | $3x - 4 = 0$, $x = \frac{4}{3}$

Step 2: Find $f''(x)$

$$f''(x) = 12x^3 - 12x^2$$
$$= 12x^2(x-1)$$

Step 3: Plug ① into ②

$$f''(0) = 12(0)^2(0-1) = 0$$

$$f''\left(\frac{4}{3}\right) = 12\left(\frac{4}{3}\right)^2\left(\frac{4}{3}-1\right)$$

Step 4: Use Second Derivative Test

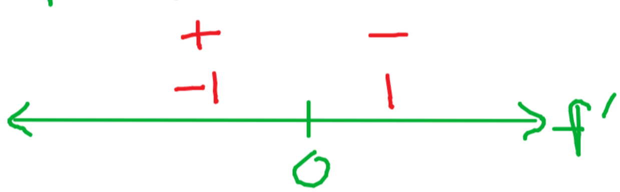
$f''(0) = 12(0)^2(0-1) = 0 \implies$ Fails so we need to use First Derivative Test

$f''\left(\frac{4}{3}\right) = 12\left(\frac{4}{3}\right)^2\left(\frac{4}{3}-1\right) > 0$ ∪

$\implies$ rel min @ $x = \frac{4}{3}$

First Derivative Test for $x = 0$.

$$f'(x) = x^3(3x-4)$$



∩ $\implies$ relative max @ $x = 0$