Lesson 23: A Summary of Curve Sketching

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Now we are going to assemble what we learned to analyze and sketch rational functions.

Ex 1: Analyze and sketch the graph of $f(x)=\frac{x^2-x}{x-3}$

① Domain of f: When does f(x) DNE?

Fractions are undefined when denominator = 0

$x-3=0$

$x=3 \implies$ Domain: $(-\infty,3)\cup(3,\infty)$

② x-intercept: Set y=0. Solve for x.

$\frac{0}{1}=\frac{x^2-x}{x-3}$

$0(x-3)=x^2-x$

$0=x^2-x$

$0=x(x-1)$

$0=x \quad | \quad 0=x-1$

$\quad\quad\quad | \quad x=1$

x-intercepts: $(0,0)$, $(1,0)$

③ y-intercept: Set x=0. Solve for y.

$y=\frac{0^2-0}{0-3}=\frac{0}{-3}=0$

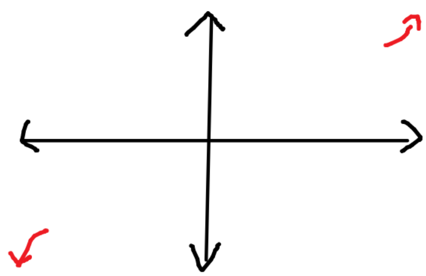
y-intercept: $(0,0)$

$$f(x)=\frac{x^2-x}{x-3}$$

④ End Behavior: Use the General Rule

ⓐ $\lim\limits_{x\to-\infty} f(x) = \lim\limits_{x\to-\infty} \frac{x^2}{x} = \lim\limits_{x\to-\infty} x = -\infty$

ⓑ $\lim\limits_{x\to+\infty} f(x) = \lim\limits_{x\to+\infty} \frac{x^2}{x} = \lim\limits_{x\to+\infty} x = +\infty$



$$f(x)=\frac{x^2-x}{x-3}=\frac{x(x-1)}{x-3}$$

⑤ Asymptotes

ⓐ Vertical Asymptote:

① Check that $f(x)$ is simplify. ✓

② Set denominator to 0. Solve.

$x-3=0$

$x=3 \Rightarrow$ VA: $x=3$

ⓑ Horizontal Asymptote: Use ④

Since $\lim\limits_{x\to\pm\infty} f(x) \neq L$, where L is a constant, then there is no HA.

ⓒ Slant Asymptote

① Check that difference b/w the powers of the leading terms of numerator and denominator is equal to 1. ✓

② If so, use Synthetic Division or Long Division.

$$\begin{array}{c|ccc} 3 & 1 & -1 & 0 \\ & \downarrow & 3 & 6 \\ \hline & 1 & 2 & 6 \end{array}$$

$\Rightarrow f(x) = \boxed{x+2} + \frac{6}{x-3}$

$\Downarrow$

Slant Asymptote @ $y = x+2$

⑥ Critical #s: $f'(x) = 0$ and $f'(x)$ DNE

$u(x) = x^2 - x$ $\quad$ $v(x) = x-3$

$u'(x) = 2x-1$ $\quad$ $v'(x) = 1$

	$2x$	-1
x	$2x^2$	$-x$
-3	$-6x$	3

$$f'(x) = \frac{(2x-1)(x-3) - (x^2-x)}{(x-3)^2}$$

$$= \frac{2x^2 - 7x + 3 - x^2 + x}{(x-3)^2}$$

$$= \frac{x^2-6x+3}{(x-3)^2} = 0$$

$x^2 - 6x + 3 = 0 \leftarrow$ Use Quadratic Formula

$$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(3)}}{2(1)}$$

$$= \frac{6 \pm \sqrt{36-12}}{2}$$

$$= \frac{6 \pm \sqrt{24}}{2} = \frac{6 \pm 2\sqrt{6}}{2} = 3 \pm \sqrt{6}$$

$f'(x) = \frac{x^2-6x+3}{(x-3)^2}$ DNE when $(x-3)^2 = 0$

$x = 3$

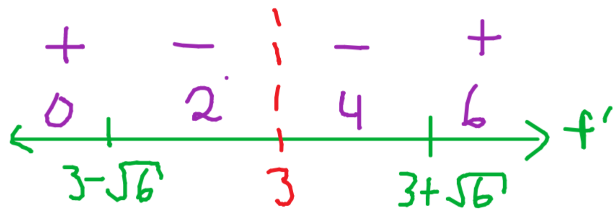
Recall domain of f is $(-\infty, 3) \cup (3, \infty)$. So $x=3$ can't be a critical #.

Critical #s: $x = 3 \pm \sqrt{6}$

$$f(x) = \frac{x^2 - x}{x-3} \qquad f'(x) = \frac{x^2 - 6x + 3}{(x-3)^2}$$

⑦ Increasing/Decreasing

Note denominator is always positive.

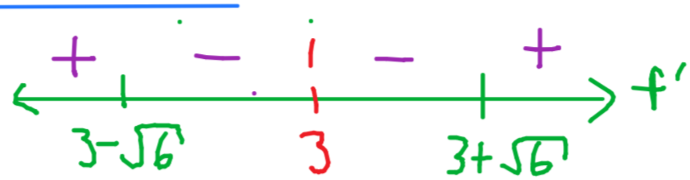


b/c f isn't defined @ $x = 3$

Increasing:
$(-\infty, 3-\sqrt{6}) \cup (3+\sqrt{6}, \infty)$

Decreasing:
$(3-\sqrt{6}, 3) \cup (3, 3+\sqrt{6})$
$= (3-\sqrt{6}, 3+\sqrt{6})$

⑧ Relative Extrema: Use ⑦



By First Derivative Test,

Rel max: $x = 3 - \sqrt{6}$

Rel min: $x = 3 + \sqrt{6}$

$$f(x)=\frac{x^2-x}{x-3} \qquad f'(x)=\frac{x^2-6x+3}{(x-3)^2}$$

9) Concavity: $f''(x)=0$ and $f''(x)$ DNE

$u(x)=x^2-6x+3 \qquad v(x)=(x-3)^2$

$u'(x)=2x-6 \qquad v'(x)=2(x-3)$

$\quad\;\; =2(x-3)$

$$f''(x)=\frac{2(x-3)(x-3)^2-2(x-3)(x^2-6x+3)}{(x-3)^4}$$

$$=\frac{2\cancel{(x-3)}\left[(x-3)^2-(x^2-6x+3)\right]}{(x-3)^{\cancel{4}\,3}}$$

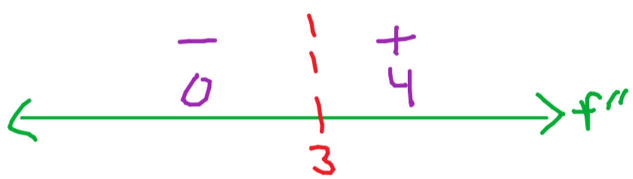
$$=\frac{2\left[\cancel{x^2}-\cancel{6x}+9-\cancel{x^2}+\cancel{6x}-3\right]}{(x-3)^3}=\frac{2\cdot 6}{(x-3)^3}=\frac{12}{(x-3)^3}=0$$

$f''(x)=\frac{12}{(x-3)^3}=0 \Rightarrow f''(x)\neq 0$ b/c numerator $\neq 0$

$f''(x)$ DNE when $(x-3)^3=0$

$x=3$

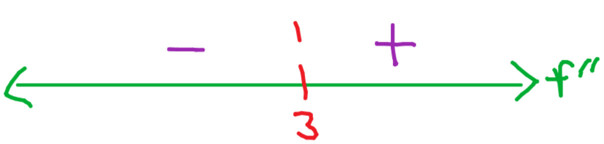
But we know $f(x)$ DNE @ $x=3$



Concave Up: $(3,\infty)$

Concave Down: $(-\infty,3)$

10) Inflection Points: Use 9 and check for sign change



Is there a change? Yes

But $f(x)$ isn't defined @ $x=3$, so no inflection points.

(11) Graph $f(x) = \frac{x^2 - x}{x - 3}$