MA 16010 LESSONS 24+25+26: OPTIMIZATION

Optimization Problems: Often this includes finding the maximum or minimum value of some function.

- e.g. The minimum time to make a certain journey,
- e.g. The minimum cost for constructing some object,
- e.g. The maximum profit to gain for a business, and so on.

How do we solve an optimization problem?

- Determine a function (**known as objective function)** that we need to maximize or minimize.

- Determine if there are some constraints on the variables. (The equations that describe the constraints are called **the constraint equations**.)
    - o If there are constraint equations, rewrite the **objective function** as a function of only one variable.

- Then we can solve for absolute maximum or minimum like we did before.
    - o Using either the First Derivative Test or Second Derivative Test.

Recipe for Solving an Optimization Problem

Step 1: Identify what quantity you are trying to optimize.

Step 2: Draw a picture (if applicable), corresponding to the problem, and label it with your variables.

Step 3: Express the variable to be optimized as a function of the variables you used in Step 2.

Step 4: Find relations among the variables from Step 2 and express the variable to be optimized a function of just one of the variables from Step 2.

Step 5: Identify the domain for the function you found in Step 4.

Step 6: Find the absolute extrema of the variable to be optimized on this domain.

Step 7: Reread the question and be sure you have answered exactly what was asked.

Let x and y be such #s.

Example 1: Of all the numbers whose sum is 50, find the two that have the maximum product.

Step 1: Identify what quantity you are trying to optimize. Product

Step 2: Draw a picture (if applicable), corresponding to the problem, and label it with your variables.

None

Step 3: Express the variable to be optimized as a function of the variables you used in Step 2.

$P = xy$

Step 4: Find relations among the variables from Step 2 and express the variable to be optimized a function of just one of the variables from Step 2.

$x + y = 50$

Step 5: Identify the domain for the function you found in Step 4.

All numbers $\Rightarrow (-\infty, \infty)$

Domain x & y: $(-\infty, \infty)$

Step 6: Find the absolute extrema of the variable to be optimized on this domain.

Solve ④ for y.

$y = 50 - x$

Plug y into ③

$P = x(50 - x)$

$= 50x - x^2$

Find P' and set = 0.

$P' = 50 - 2x = 0$

$50 = 2x$

$x = 25$

Check x=25 gives abs max. By 2nd Derivative Test,

$P'' = -2$

$P''(25) = -2 < 0$

$\Rightarrow$ abs max

Step 7: Reread the question and be sure you have answered exactly what was asked.

$\boxed{x = 25} \Rightarrow y = 50 - x$

$\boxed{y = 25}$

<u>Example 2:</u> A carpenter is building a rectangular room with a fixed perimeter of 54 feet. What are the dimensions of the largest room that can be built? What is its area?

Step 1: Identify what quantity you are trying to optimize. Area, A

Step 2: Draw a picture (if applicable), corresponding to the problem, and label it with your variables.

Step 3: Express the variable to be optimized as a function of the variables you used in Step 2.

$$A = xy$$

Step 4: Find relations among the variables from Step 2 and express the variable to be optimized a function of just one of the variables from Step 2.

$$54 = P = 2x + 2y \iff 27 = x + y$$

Step 5: Identify the domain for the function you found in Step 4.

no length $\Rightarrow x = 0, y = 0$
no width $\Rightarrow x = 27, y = 27$
Domain: x & y: $(0, 27)$

Step 6: Find the absolute extrema of the variable to be optimized on this domain.

Solve ④ for y
$y = 27 - x$
Plug y into ③
$A = x(27 - x)$
$= 27x - x^2$

Find A' and set = 0.
$A' = 27 - 2x = 0$
$27 = 2x$
$x = \frac{27}{2}$

Check $x = 27/2$ gives abs max. By 2nd Derivative Test,
$A'' = -2$
$A''(\frac{27}{2}) = -2 < 0$
$\Rightarrow$ abs max

Step 7: Reread the question and be sure you have answered exactly what was asked.

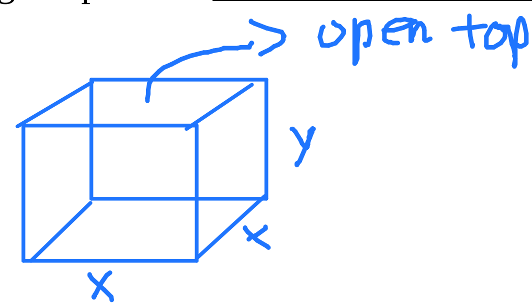
$x = \frac{27}{2} \Rightarrow y = 27 - x$, $y = \frac{27}{2}$

Area $= \frac{27}{2} \times \frac{27}{2} = \frac{749}{4} = A$

Example 3: An open-top box with a square base is to have a volume of 8 cubic feet. Find the dimensions of the box that can be made with the least amount of material.

Step 1: Identify what quantity you are trying to optimize. Surface Area, A

Step 2: Draw a picture (if applicable), corresponding to the problem, and label it with your variables.



Step 3: Express the variable to be optimized as a function of the variables you used in Step 2.

$$A = x^2 + 4xy$$

Step 4: Find relations among the variables from Step 2 and express the variable to be optimized a function of just one of the variables from Step 2.

$$8 = V = x^2 y$$

Step 5: Identify the domain for the function you found in Step 4.

no length $\Rightarrow x=0, y=0$

no width $\Rightarrow 8 = 0^2 \cdot y$

$8 = 0 \Rightarrow$ only big

Domain: $(0, \infty)$ x & y

Step 6: Find the absolute extrema of the variable to be optimized on this domain.

Solve ④ for y

$$y = \frac{8}{x^2}$$

Plug y into ③

$$A = x^2 + 4x\left(\frac{8}{x^2}\right) = x^2 + \frac{32}{x} = x^2 + 32x^{-1}$$

Find A′ and set = 0.

$$A' = 2x - 32x^{-2} = 0$$

$$2x - \frac{32}{x^2} = 0$$

$$\frac{2x}{1} = \frac{32}{x^2}$$

$$2x^3 = 32$$

$$x^3 = 16$$

$$x = \sqrt[3]{16}$$

Check $x = \sqrt[3]{16}$ gives abs min. By 2nd Derivative Test,

$$A'' = 2 - 32(-2)x^{-3} = 2 + \frac{64}{x^3}$$

$$A''(\sqrt[3]{16}) = 2 + \frac{64}{16} > 0$$

$\Rightarrow$ abs min

Step 7: Reread the question and be sure you have answered exactly what was asked.

$$x = \sqrt[3]{16} \Rightarrow y = \frac{8}{x^2} = \frac{8}{16^{2/3}}$$

Dimensions

$$\sqrt[3]{16} \times \sqrt[3]{16} \times \frac{8}{16^{2/3}}$$

Example 4: From a thin piece of cardboard 20 in by 20 in, square corners are cut out so that the sides can be folded up to make a box. What dimensions will yield a box of maximum volume? What is the maximum volume?

Step 1: Identify what quantity you are trying to optimize. ______________________________

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Step 7: Reread the question and be sure you have answered exactly what was asked.

Example 5: A company's marketing department has determined that if their product is sold at the price of p dollars per unit, they can sell $q = 2400 - 200p$ units. Each unit costs \$5 to make.

1. What price, p, should the company charge to maximize their revenue?

Step 1: Identify what quantity you are trying to optimize. ______________________________

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Step 6: Find the absolute extrema of the variable to be optimized on this domain.

Step 7: Reread the question and be sure you have answered exactly what was asked.

<u>Example 5:</u> A company's marketing department has determined that if their product is sold at the price of p dollars per unit, they can sell $q = 2400 - 200p$ units. Each unit costs \$5 to make.

2. What price, p, should the company charge to maximize their profit?

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Step 7: Reread the question and be sure you have answered exactly what was asked.

Example 6: A Norman window is constructed by adjoining a semicircle to the top of an ordinary rectangular window. Find the dimensions of a Norman window of maximum area if the total perimeter is 10 feet.

Example 7: A rectangular box is to have a square base and a volume of 800 ft^3. If the material for the base costs \$2 per square foot, the material for the sides costs \$4 per square foot, and the material for the top costs \$1 per square foot, determine the minimum cost for constructing such a box.

Example 8: Find the point on the graph of $f(x) = 2x + 4$ that is the closest to the point $(1,3)$.