MA 16010 LESSONS 24+25+26: OPTIMIZATION

Optimization Problems: Often this includes finding the maximum or minimum value of some function.

- e.g. The minimum time to make a certain journey,
- e.g. The minimum cost for constructing some object,
- e.g. The maximum profit to gain for a business, and so on.

How do we solve an optimization problem?

- Determine a function (**known as objective function)** that we need to maximize or minimize.

- Determine if there are some constraints on the variables. (The equations that describe the constraints are called **the constraint equations**.)
  - o If there are constraint equations, rewrite the **objective function** as a function of only one variable.

- Then we can solve for absolute maximum or minimum like we did before.
  - o Using either the First Derivative Test or Second Derivative Test.

Recipe for Solving an Optimization Problem

Step 1: Identify what quantity you are trying to optimize.

Step 2: Draw a picture (if applicable), corresponding to the problem, and label it with your variables.

Step 3: Express the variable to be optimized as a function of the variables you used in Step 2.

Step 4: Find relations among the variables from Step 2 and express the variable to be optimized a function of just one of the variables from Step 2.

Step 5: Identify the domain for the function you found in Step 4.

Step 6: Find the absolute extrema of the variable to be optimized on this domain.

Step 7: Reread the question and be sure you have answered exactly what was asked.

Let x and y be such #s.

Example 1: Of all the numbers whose sum is 50, find the two that have the maximum product.

Step 1: Identify what quantity you are trying to optimize. Product

Step 2: Draw a picture (if applicable), corresponding to the problem, and label it with your variables.

None

Step 3: Express the variable to be optimized as a function of the variables you used in Step 2.

$P = xy$

Step 4: Find relations among the variables from Step 2 and express the variable to be optimized a function of just one of the variables from Step 2.

$x + y = 50$

Step 5: Identify the domain for the function you found in Step 4.

All numbers $\Rightarrow (-\infty, \infty)$

Domain x & y: $(-\infty, \infty)$

Step 6: Find the absolute extrema of the variable to be optimized on this domain.

Solve ④ for y.

$y = 50 - x$

Plug y into ③

$P = x(50 - x)$

$= 50x - x^2$

Find P' and set $= 0$.

$P' = 50 - 2x = 0$

$50 = 2x$

$x = 25$

Check $x = 25$ gives abs max. By 2nd Derivative Test,

$P'' = -2$

$P''(25) = -2 < 0$

$\Rightarrow$ abs max

Step 7: Reread the question and be sure you have answered exactly what was asked.

$\boxed{x = 25} \Rightarrow y = 50 - x$

$\boxed{y = 25}$

<u>Example 2:</u> A carpenter is building a rectangular room with a fixed perimeter of 54 feet. What are the dimensions of the largest room that can be built? What is its area?

Step 1: Identify what quantity you are trying to optimize. Area, A

Step 2: Draw a picture (if applicable), corresponding to the problem, and label it with your variables.

Step 3: Express the variable to be optimized as a function of the variables you used in Step 2.

$$A = xy$$

Step 4: Find relations among the variables from Step 2 and express the variable to be optimized a function of just one of the variables from Step 2.

$$54 = P = 2x + 2y \iff 27 = x + y$$

Step 5: Identify the domain for the function you found in Step 4.

$$\left.\begin{array}{l} \text{no length} \Rightarrow x = 0, y = 0 \\ \text{no width} \Rightarrow x = 27, y = 27 \end{array}\right\} \begin{array}{l} \text{Domain} \\ x \,\&\, y \end{array} : (0, 27)$$

Step 6: Find the absolute extrema of the variable to be optimized on this domain.

Solve ④ for y

$y = 27 - x$

Plug y into ③

$A = x(27 - x)$

$= 27x - x^2$

Find A' and set $= 0$.

$A' = 27 - 2x = 0$

$27 = 2x$

$x = \frac{27}{2}$

Check $x = 27/2$ gives abs max. By 2^{nd} Derivative Test,

$A'' = -2$

$A''\left(\frac{27}{2}\right) = -2 < 0$

$\Rightarrow$ abs max

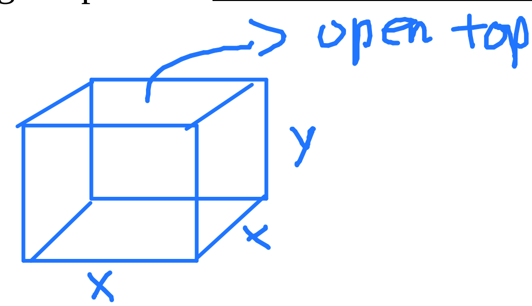
Step 7: Reread the question and be sure you have answered exactly what was asked.

$$\left.\begin{array}{l} \boxed{x = \frac{27}{2}} \Rightarrow y = 27 - x \\ \qquad\qquad \boxed{y = \frac{27}{2}} \end{array}\right\} \text{Area} = \frac{27}{2} \times \frac{27}{2} = \boxed{\frac{749}{4} = A}$$

Example 3: An open-top box with a square base is to have a volume of 8 cubic feet. Find the dimensions of the box that can be made with the least amount of material.

Step 1: Identify what quantity you are trying to optimize. Surface Area, A

Step 2: Draw a picture (if applicable), corresponding to the problem, and label it with your variables.



Step 3: Express the variable to be optimized as a function of the variables you used in Step 2.

$$A = x^2 + 4xy$$

Step 4: Find relations among the variables from Step 2 and express the variable to be optimized a function of just one of the variables from Step 2.

$$8 = V = x^2 y$$

Step 5: Identify the domain for the function you found in Step 4.

no length $\Rightarrow x=0, y=0$

no width $\Rightarrow 8 = 0^2 \cdot y$

$8 = 0 \Rightarrow$ ∞ly big

Domain: $(0, \infty)$ x & y

Step 6: Find the absolute extrema of the variable to be optimized on this domain.

Solve ④ for y

$y = \frac{8}{x^2}$

Plug y into ③

$A = x^2 + 4x\left(\frac{8}{x^2}\right)$

$= x^2 + \frac{32}{x}$

$= x^2 + 32x^{-1}$

Find A' and set $= 0$.

$A' = 2x - 32x^{-2} = 0$

$2x - \frac{32}{x^2} = 0$

$\frac{2x}{1} = \frac{32}{x^2}$

$2x^3 = 32$

$x^3 = 16$

$x = \sqrt[3]{16}$

Check $x = \sqrt[3]{16}$ gives abs min. By 2nd Derivative Test,

$A'' = 2 - 32(-2)x^{-3}$

$= 2 + \frac{64}{x^3}$

$A''(\sqrt[3]{16}) = 2 + \frac{64}{16} > 0$

$\Rightarrow$ abs min

Step 7: Reread the question and be sure you have answered exactly what was asked.

$x = \sqrt[3]{16} \Rightarrow y = \frac{8}{x^2} = \frac{8}{16^{2/3}}$

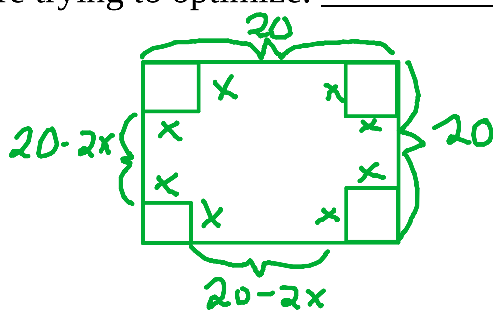
Dimensions

$\boxed{\sqrt[3]{16} \times \sqrt[3]{16} \times \frac{8}{16^{2/3}}}$

Example 4: From a thin piece of cardboard 20 in by 20 in, square corners are cut out so that the sides can be folded up to make a box. What dimensions will yield a box of maximum volume? What is the maximum volume?

Step 1: Identify what quantity you are trying to optimize. Volume

Step 2: Draw a picture (if applicable), corresponding to the problem, and label it with your variables.



Step 3: Express the variable to be optimized as a function of the variables you used in Step 2.

$$V = x(20-2x)^2$$

Step 4: Find relations among the variables from Step 2 and express the variable to be optimized a function of just one of the variables from Step 2.

None since V is in terms of x only.

Step 5: Identify the domain for the function you found in Step 4.

no length $\Rightarrow x = 0$

no width $\Rightarrow 20-2x = 0$

$x = 10$

Domain of x: $(0, 10)$

Step 6: Find the absolute extrema of the variable to be optimized on this domain.

Find V' and set $= 0$.

$$V' = (20-2x)^2 + x \cdot 2(20-2x)(-2) = 0$$

$$(20-2x)[20-2x-4x] = 0$$

$$(20-2x)(20-6x) = 0$$

$$\cancel{x = 10} \qquad x = \frac{10}{3}$$

Note $x = 10$ isn't in our domain.

So only check $x = 10/3$ gives abs max.

$$V'' = -2(20-6x) + (-6)(20-2x)$$

$$V''\left(\tfrac{10}{3}\right) = -2\left(\cancel{20-6\left(\tfrac{10}{3}\right)}^{0}\right) - 6\left(20-2\left(\tfrac{10}{3}\right)\right) < 0$$

$\Rightarrow$ abs max

Step 7: Reread the question and be sure you have answered exactly what was asked.

$x = \frac{10}{3} \Rightarrow w = 20-2x$

$= \frac{40}{3}$

$$\boxed{\begin{array}{l}\underline{\text{Dimensions}}: \frac{10}{3} \times \frac{40}{3} \times \frac{40}{3} \\ \underline{\text{Volume}}: V = 16000/27\end{array}}$$

Example 5: A company's marketing department has determined that if their product is sold at the price of p dollars per unit, they can sell $q = 2400 - 200p$ units. Each unit costs \$5 to make.

1. What price, p, should the company charge to maximize their revenue?

Step 1: Identify what quantity you are trying to optimize. Revenue

Step 2: Draw a picture (if applicable), corresponding to the problem, and label it with your variables.

None

Step 3: Express the variable to be optimized as a function of the variables you used in Step 2.

$R = pq$

Step 4: Find relations among the variables from Step 2 and express the variable to be optimized a function of just one of the variables from Step 2.

$q = 2400 - 200p$

Step 5: Identify the domain for the function you found in Step 4.

cheapest $p = 0$ (freebies)

$\Rightarrow$ Domain of p is $[0, \infty)$

Step 6: Find the absolute extrema of the variable to be optimized on this domain.

Plug q into ③.

$R = p(2400 - 200p)$

$= 2400p - 200p^2$

Find R' and set $= 0$.

$R' = 2400 - 400p = 0$

$p = 6$

Using $p = 0, 6$ we check for abs max.

$R(0) = 0(2400) = 0$

$R(6) = 6(2400 - 200(6))$

$= 6(1200) = 7200$

Hence $p = 6$ is abs max.

Step 7: Reread the question and be sure you have answered exactly what was asked.

$p = \$6$

Example 5: A company's marketing department has determined that if their product is sold at the price of p dollars per unit, they can sell $q = 2400 - 200p$ units. Each unit costs \$5 to make.

2. What price, p, should the company charge to maximize their profit?

Step 1: Identify what quantity you are trying to optimize. ______________________________

Step 2: Draw a picture (if applicable), corresponding to the problem, and label it with your variables.

None

Step 3: Express the variable to be optimized as a function of the variables you used in Step 2.

$P = R - C = pq - 5q = (p-5)q$

Step 4: Find relations among the variables from Step 2 and express the variable to be optimized a function of just one of the variables from Step 2.

$q = 2400 - 200p$

Step 5: Identify the domain for the function you found in Step 4.

cheapest $p = 0$ (freebies)

$\Rightarrow$ Domain of p is $[0, \infty)$

Step 6: Find the absolute extrema of the variable to be optimized on this domain.

Plug ④ into ③.

$P = (p-5)(2400-200p)$

$= -200(p-5)(p-12)$

$= -200(p^2 - 17p + 60)$

Find P' and set $= 0$.

$P' = -200(2p - 17) = 0$

$p = 17/2 = 8.5$

Using $p = 0, 8.5$ we check for abs max.

$P(0) = -200(-5)(-12) = -12000$

$P(8.5) = -200(8.5-5)(8.5-12)$

$= -200(3.5)(-3.5) = 2450$

Hence $p = 8.5$ is abs max.

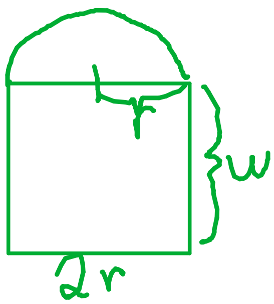
Step 7: Reread the question and be sure you have answered exactly what was asked.

$p = \$8.50$

Example 6: A Norman window is constructed by adjoining a semicircle to the top of an ordinary rectangular window. Find the dimensions of a Norman window of maximum area if the total perimeter is 10 feet.

① Area

②



③ $A = \frac{1}{2}\pi r^2 + 2rw$

④ $10 = P = 2w + 2r + \pi r$

⑤ If $r = 0$,

$10 = 2w$

$w = 5$

If $w = 0$,

$10 = 2r + \pi r$

$10 = r(2 + \pi)$

$r = \frac{10}{2+\pi}$

Domain of r: $\left(0, \frac{10}{2+\pi}\right)$

Domain of w: $(0, 5)$

⑥ Solve ④ for w.

$10 - 2r - \pi r = 2w$

$\frac{10 - 2r - \pi r}{2} = w$

Find A' and set = 0.

$A' = \pi r + 10 - 4r - 2\pi r = 0$

$10 = -\pi r + 4r + 2\pi r$

$10 = 4r + \pi r$

$10 = r(4 + \pi)$

$r = \frac{10}{4+\pi}$

Check $r = \frac{10}{4+\pi}$ gives an abs max.

$A'' = \pi - 4 - 2\pi$

$= -4 - \pi$

$A''\left(\frac{10}{4+\pi}\right) = -4 - \pi < 0$

$\Rightarrow$ abs max

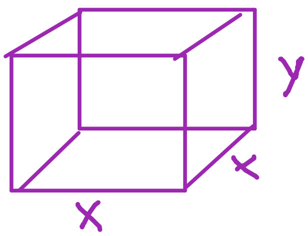
⑦ $r = \frac{10}{4+\pi}$

$w = \frac{10 - 2r - \pi r}{2} = \frac{10}{4+\pi}$

Example 7: A rectangular box is to have a square base and a volume of 800 ft^3. If the material for the base costs \$2 per square foot, the material for the sides costs \$4 per square foot, and the material for the top costs \$1 per square foot, determine the minimum cost for constructing such a box.

① Cost, C

②



③ $C = 2x^2 + 4(4xy) + x^2$

$= 3x^2 + 16xy$

④ $800 = x^2 y$

⑤ Domain of x & y:

$(0, \infty)$

⑥ Solve ④ for y.

$y = \frac{800}{x^2}$

Plug y into ③

$C = 3x^2 + 16x\left(\frac{800}{x^2}\right)$

$= 3x^2 + \frac{12800}{x}$

$= 3x^2 + 12800x^{-1}$

Find C' and set = 0.

$C' = 6x - 12800x^{-2} = 0$

$6x - \frac{12800}{x^2} = 0$

$\frac{6x}{1} = \frac{12800}{x^2}$

$6x^3 = 12800$

$x^3 = 6400/3$

$x = \sqrt[3]{\frac{6400}{3}}$

Check x gives an abs min.

$C'' = 6 - 12800(-2)x^{-3}$

$= 6 + \frac{25600}{x^3}$

$C''\left(\sqrt[3]{\frac{6400}{3}}\right) = 6 + \frac{25600}{6400/3} > 0$

$\Rightarrow$ abs min

⑦ $x = \sqrt[3]{\frac{6400}{3}}$

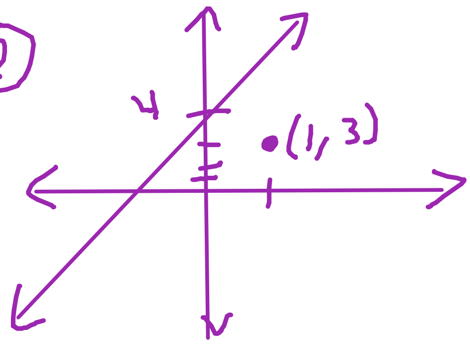
$y = \frac{800}{\left(\frac{6400}{3}\right)^{2/3}}$

$\boxed{C = \$62.14}$

Example 8: Find the point on the graph of $f(x) = 2x + 4$ that is the closest to the point $(1,3)$.

① Distance, D

②



③ $D = (x-1)^2 + (y-3)^2$

④ $y = f(x) = 2x + 4$

⑤ From the graph, x, y's domain is $(-\infty, \infty)$

⑥ Plug ④ into ③.

$D = (x-1)^2 + (2x+4-3)^2$

$= (x-1)^2 + (2x+1)^2$

Find D' and set $= 0$.

$D' = 2(x-1) + 2(2x+1) \cdot 2 = 0$

$2x - 2 + 4(2x+1) = 0$

$2x - 2 + 8x + 4 = 0$

$10x + 2 = 0$

$x = -\frac{1}{5}$

Check $x = -\frac{1}{5}$ gives abs min.

$D'' = 10$

$D''(-1/5) = 10 > 0$

$\Rightarrow$ abs min

⑦ $x = -\frac{1}{5}$

$y = 2x + 4 = \frac{18}{5}$

$\boxed{\left(-\frac{1}{5}, \frac{18}{5}\right)}$