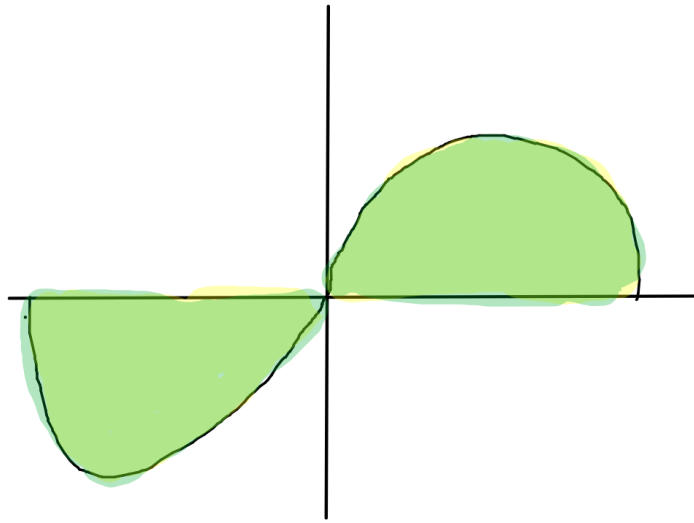


# Lesson 29: Area and Riemann Sums

## Lesson 29: Area and Riemann Sums



Purpose: How to approx the signed area under the curve of a function

Ex 1: Evaluate  $\sum_{i=1}^5 i^2$

$$= 1^2 + 2^2 + 3^2 + 4^2 + 5^2 = \textcircled{55}$$

Ex 2: Use the sigma notation to write the sum

$$2(1^3 + 1) + 2(2^3 + 1) + \dots + 2(n^3 + 1)$$

$$= \sum_{i=1}^n 2(i^3 + 1)$$

Ex 3: Use the sigma notation to write the sum

$$\frac{2}{0+5} + \frac{2}{1+5} + \dots + \frac{2}{n+5}$$

$$= \sum_{i=0}^n \frac{2}{i+5}$$

**Signed Area** is the area enclosed by the function and the x-axis with a sign.

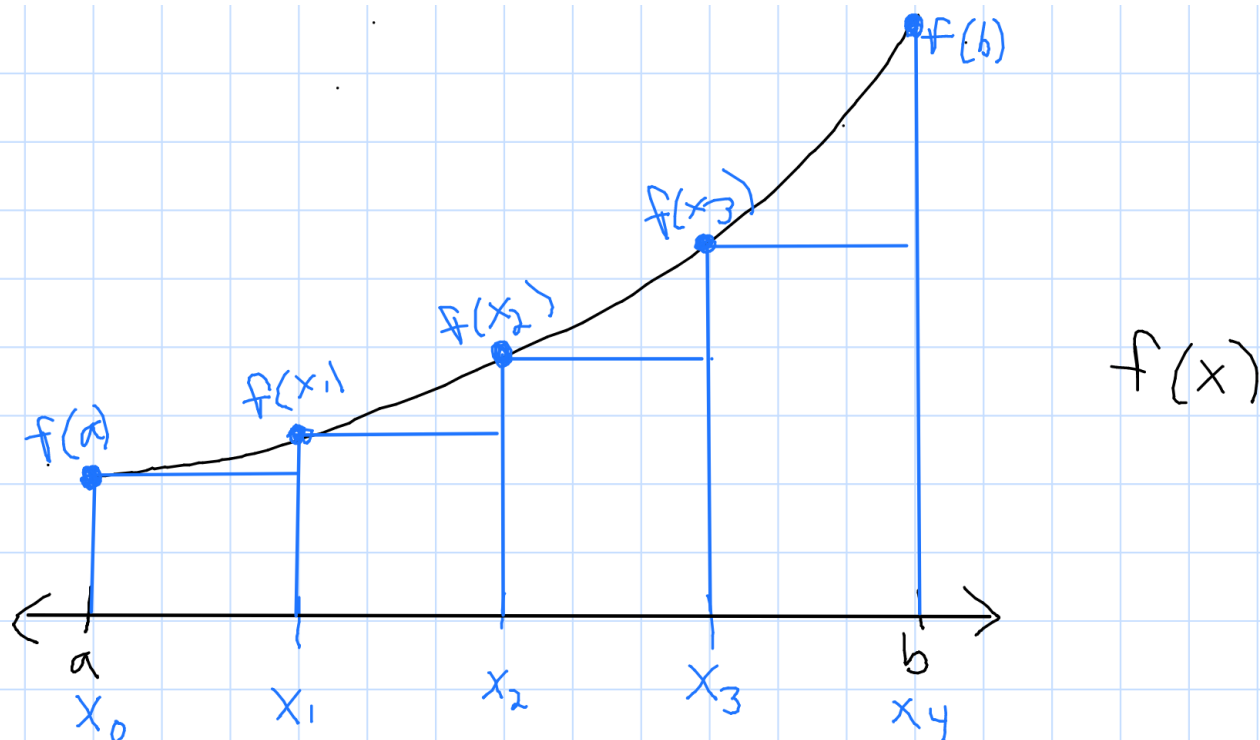
If the function is **above** the x-axis, then the area is **positive**.

If the function is **below** the x-axis, then the area is **negative**.

We want to estimate the area under  $f(x)$  from  $a$  to  $b$  by using 4 rectangles.

We divide the interval  $[a, b]$  into 4 sub-intervals.

Then construct the rectangles by starting at the pts on  $f(x)$  whose x coordinates are the left end of each of the sub-intervals.



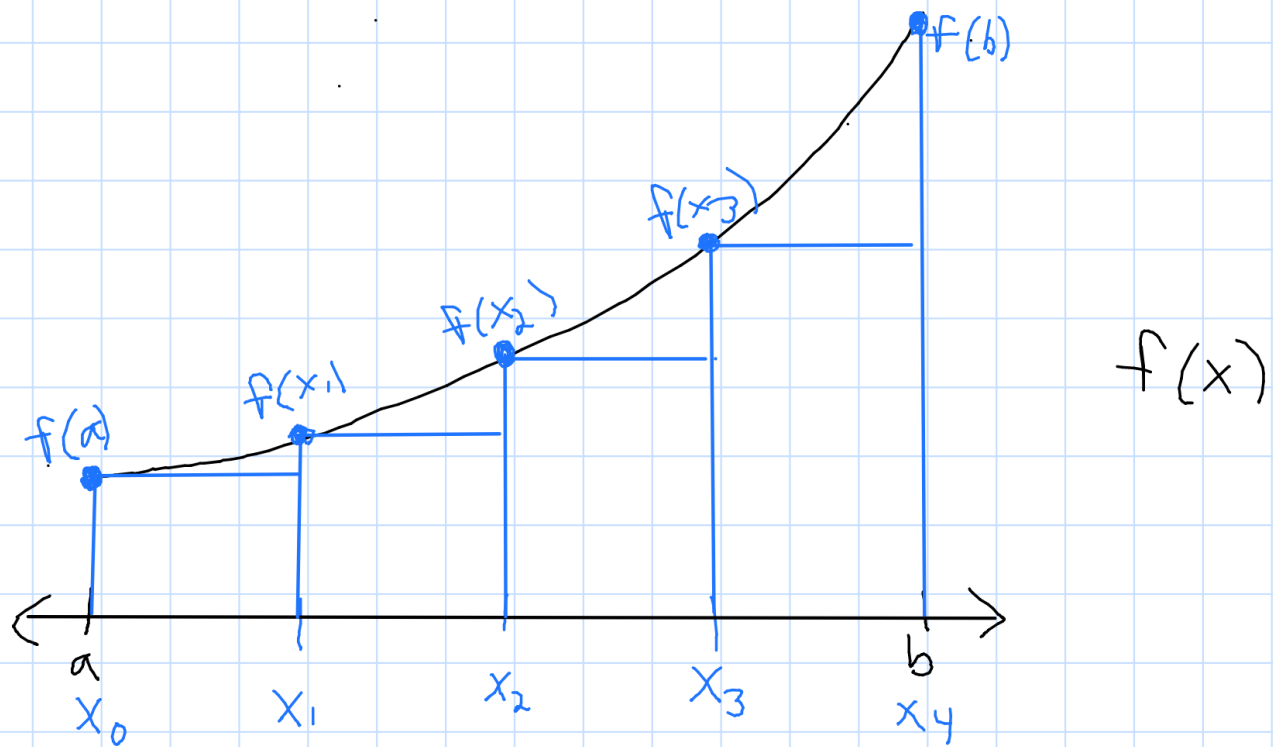
The total area of these rectangles gives us an estimate of the area under  $f(x)$ .

A sum like this is called the **Left Riemann Sum**.

How do we compute this sum?

Find the area of each rectangle and sum them all up.

But let's try to work smart



Width of each rectangle is the same.

$$w = \Delta x = \frac{b-a}{4}$$

Length of first rectangle  $f(a) = f(x_0)$   
 second rectangle  $f(x_1)$

Length of ith rectangle  $f(x_i)$

Area of each rectangle is  
 $lw = \Delta x f(x_i)$

4 rectangles  $\Rightarrow \sum_{i=0}^3 f(x_i) \Delta x$  where  $x_i = a + i \Delta x$   
 $\Delta x = \frac{b-a}{n}$

Why 3 not 4?

Then construct the rectangles by starting at the pts on  $f(x)$  whose  $x$  coordinates are the left end of each of the sub-intervals.

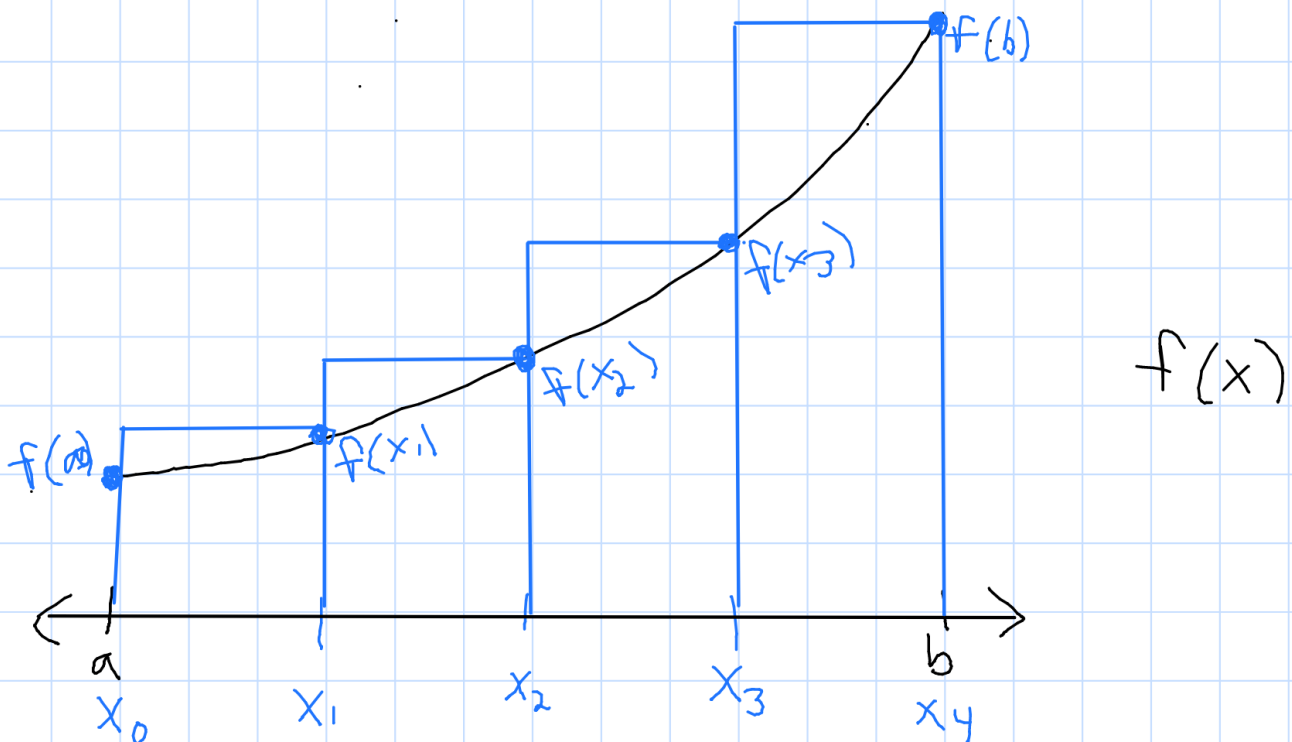
Let's devise a formula for  $n$  rectangles.

$$L_n = \sum_{i=0}^{n-1} f(x_i) \Delta x \quad \text{where } x_i = a + i \Delta x$$

$$\Delta x = \frac{b-a}{n}$$

Can we do this by using the right end of each sub-interval?

Yep It's called Right Riemann Sum



$$R_n = \sum_{i=1}^n f(x_i) \Delta x \quad \text{where } x_i = a + i \Delta x$$
$$\Delta x = \frac{b-a}{n}$$

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$$\Delta x = \frac{b-a}{n}$$

Note the only difference is the index.

Ex 4: Use the Left Riemann Sum with 3 rectangles to estimate the area under the curve  $y=x^2$  on the interval of  $[1, 7]$ .

$$a = 1 \quad n = 3$$

$$b = 7$$

$$\Delta x = \frac{b-a}{n} = \frac{7-1}{3} = \frac{6}{3} = 2$$

$$x_i = a + i\Delta x = 1 + 2i$$

$$f(x_i) = (1+2i)^2 = 1 + 4i + 4i^2$$

$$L_n = \sum_{i=0}^{n-1} f(x_i) \Delta x$$

$$L_3 = \sum_{i=0}^{3-1} (4i^2 + 4i + 1) \cdot 2$$

$$= 2 \sum_{i=0}^2 (4i^2 + 4i + 1)$$

$$= 2(4(0)^2 + 4(0) + 1) + 2(4(1)^2 + 4(1) + 1)$$

$$+ 2(4(2)^2 + 4(2) + 1)$$

$$= \textcircled{70}$$

$$\Delta x = 2$$

$$x_i = 1 + 2i$$

$$f(x_i) = 4i^2 + 4i + 1$$

$$n = 3$$

Ex 5: Use the Right Riemann Sum with 4 rectangles to estimate the area under the curve  $y = \sqrt{x} - x$  on the interval of  $[0, 4]$ .

$$a = 0 \quad n = 4$$

$$b = 4$$

$$\Delta x = \frac{b-a}{n} = \frac{4-0}{4} = 1$$

$$x_i = a + i\Delta x = 0 + i(1) = i$$

$$f(x_i) = \sqrt{i} - i$$

$$R_n = \sum_{i=1}^n f(x_i) \Delta x$$

$$\Delta x = 1$$

$$x_i = i$$

$$f(x_i) = \sqrt{i} - i$$

$$n = 4$$

$$R_4 = \sum_{i=1}^4 [\sqrt{i} - i]$$

$$= (\cancel{\sqrt{1}} - \cancel{1}) + (\sqrt{2} - \cancel{2}) + (\sqrt{3} - 3) + (\cancel{\sqrt{4}} - 4)$$

$$= \sqrt{2} + \sqrt{3} - 7$$



Ex 6: Use the Right Riemann Sum with 50 rectangles to estimate the area under the curve  $y = x^2 - 2x + 1$  on the interval of  $[0, 10]$ . Write your answer using sigma notation.

$$a = 0$$

$$b = 10$$

$$n = 50$$

$$\Delta x = \frac{b-a}{n} = \frac{10-0}{50} = \frac{10}{50} = \frac{1}{5}$$

$$x_i = a + i \Delta x = 0 + i \cdot \frac{1}{5} = i/5$$

$$f(x_i) = \left(\frac{i}{5}\right)^2 - 2\left(\frac{i}{5}\right) + 1$$

$$R_n = \sum_{i=1}^n f(x_i) \Delta x$$

$$\Delta x = \frac{1}{5}$$

$$x_i = i/5$$

$$f(x_i) = \left(\frac{i}{5}\right)^2 - 2\left(\frac{i}{5}\right) + 1$$

$$n = 50$$

$$R_{50} = \sum_{i=0}^{50} \left[ \left(\frac{i}{5}\right)^2 - 2\left(\frac{i}{5}\right) + 1 \right] \cdot \frac{1}{5}$$