Lesson 34: Numerical Integration

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With the FTC, we are able to evaluate definite integrals for certain integrands.

However, there are many functions that we do not know how to integrate

ex. ① $f(x) = e^x\sqrt{x^2+1}$

② $f(x) = \dfrac{\sin x}{x+1}$

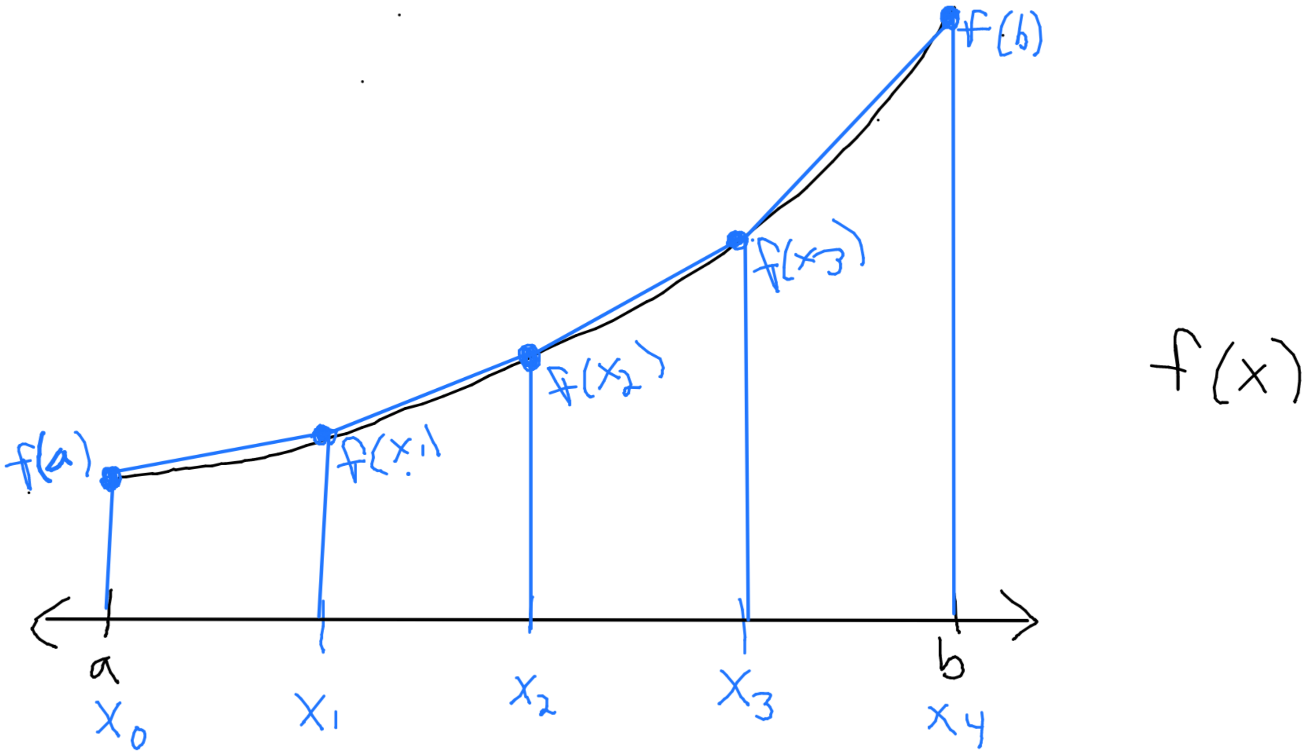
The Trapezoid Rule is an approx method that allows us to approx definite integrals.

Trapezoid Rule is similar to Riemann Sums

Instead of using rectangles, we are using trapezoids.

Suppose $f(x)$ is continuous on $[a,b]$. We want to approx the area

$\int_a^b f(x)dx$ using 4 trapezoids



Recall the area of a trapezoid is

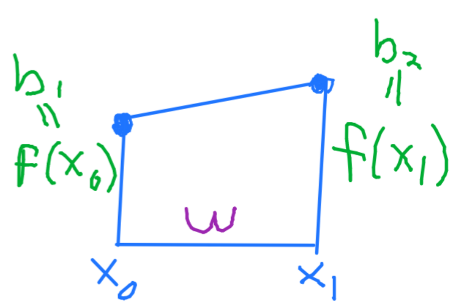
$$A=\frac{1}{2}(b_1+b_2)w$$

Good news: The width of each sub-intervals is the same as in Riemann sums

$$\Delta x=\frac{b-a}{n}=w$$

Let's determine the bases of the first trapezoid

$b_1=f(x_0)$ and $b_2=f(x_1)$



Area of 1st trapezoid, t_1, is

$$t_1=\frac{1}{2}(f(x_0)+f(x_1))\Delta x$$

Similarly,

$$t_2 = \frac{1}{2}(f(x_1) + f(x_2))\Delta x$$

$$t_3 = \frac{1}{2}(f(x_2) + f(x_3))\Delta x$$

$$t_4 = \frac{1}{2}(f(x_3) + f(x_4))\Delta x$$

Let's sum all four of these.

$$\begin{aligned} t_1 + t_2 + t_3 + t_4 = & \frac{1}{2}(f(x_0) + f(x_1))\Delta x \\ & + \frac{1}{2}(f(x_1) + f(x_2))\Delta x \\ & + \frac{1}{2}(f(x_2) + f(x_3))\Delta x \\ & + \frac{1}{2}(f(x_3) + f(x_4))\Delta x \end{aligned}$$

$$= \frac{1}{2}\begin{pmatrix} f(x_0) + f(x_1) + f(x_1) + f(x_2) \\ + f(x_2) + f(x_3) + f(x_3) + f(x_4) \end{pmatrix}\Delta x$$

$$= \frac{1}{2}(f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4))\Delta x$$

We can extend this analysis to n trapezoids.

$$T_n = \frac{1}{2}\Delta x(f(x_0) + 2f(x_1) + \ldots + 2f(x_{n-1}) + f(x_n))$$

Where $x_i = a + i\Delta x$

$\Delta x = \frac{b-a}{n}$

MA 16010 LESSON 34: NUMERICAL INTEGRATION (Examples w/ Solutions)

EX 1: Use the Trapezoid Rule to approximate $\int_0^3 x^2\ dx$ using $n = 3$. Round your answer to the nearest tenth.

Solution: (1) First calculate Δx.

a = 0

b = 3

$b - a$ = $3-0=3$

$\Delta x = \dfrac{b-a}{n}$ = $\frac{3}{3}=1$

(2) Determine what $f(x)$ is.

$$\int_0^3 \boxed{x^2}\ dx$$

Hence $f(x) =$ x^2

(3) Find the following values:

x_0 = 0		$f(x_0)$ = 0		$f(x_0)$ = 0	
x_1 = 1		$f(x_1)$ = 1		$2 \cdot f(x_1)$ = 2	
x_2 = 2		$f(x_2)$ = 4		$2 \cdot f(x_2)$ = 8	
x_3 = 3		$f(x_3)$ = 9		$f(x_3)$ = 9	

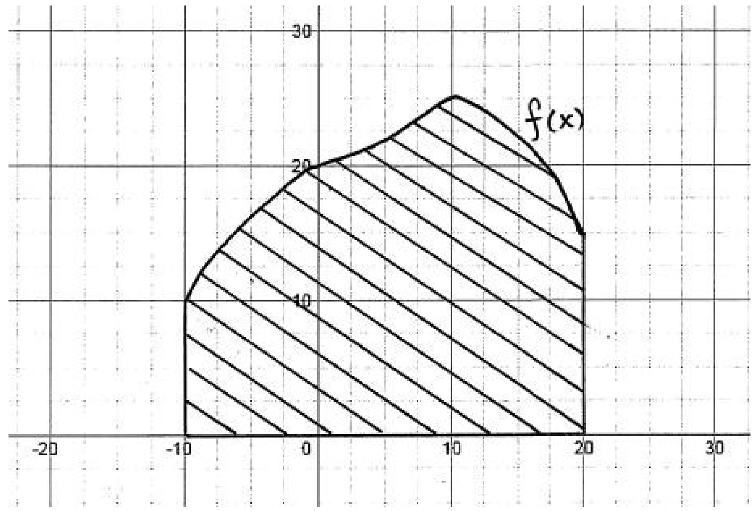
(4) Sum all the values in the black box. = 19

(5) Multiply the value found in (4), Δx found in (1), and 1/2, which yields our answer.

$19 \times 1 \times \frac{1}{2} = \frac{19}{2}$

MA 16010 LESSON 34: NUMERICAL INTEGRATION
(Examples w/ Solutions)

EX 2: Approximate the area of the shaded region by using the Trapezoid Rule with $n = 3$



Solution: (1) First calculate Δx.

a = −10

b = 20

$b - a$ = 20−(−10)=30

$\Delta x = \dfrac{b-a}{n}$ = $\frac{30}{3}$=10

(2) Find the following values:

x_0 = −10 $f(x_0)$ = 10 $f(x_0)$ = 10

x_1 = 0 $f(x_1)$ = 20 $2 \cdot f(x_1)$ = 40

x_2 = 10 $f(x_2)$ = 25 $2 \cdot f(x_2)$ = 50

x_3 = 20 $f(x_3)$ = 15 $f(x_3)$ = 15

(3) Sum all the values in the black box. = 115

(4) Multiply the value found in (4), Δx found in (1), and 1/2, which yields our answer.

$115 \times 10 \times \frac{1}{2} = 575$