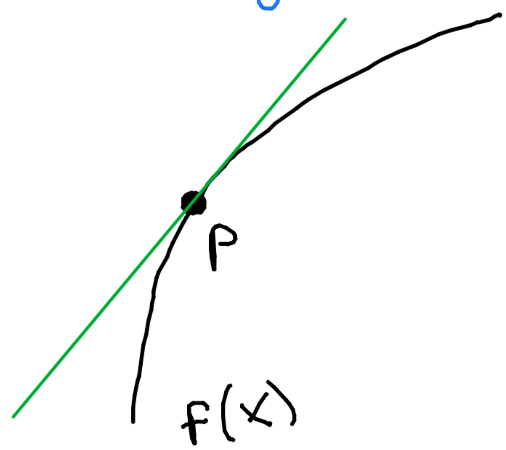
Lesson 6: The Derivative

Lesson 6. The Derivative

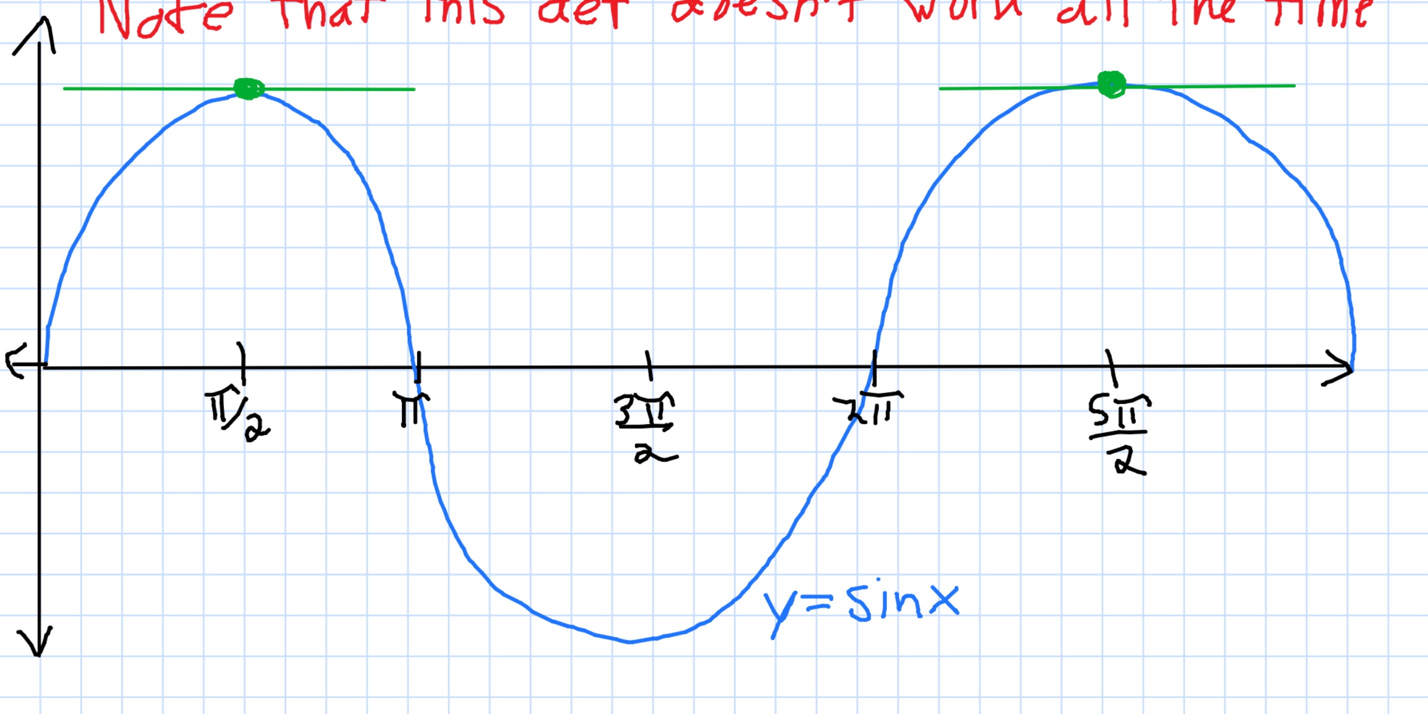
Tangent Lines are Important in Calculus!!!

What is a Tangent Line?

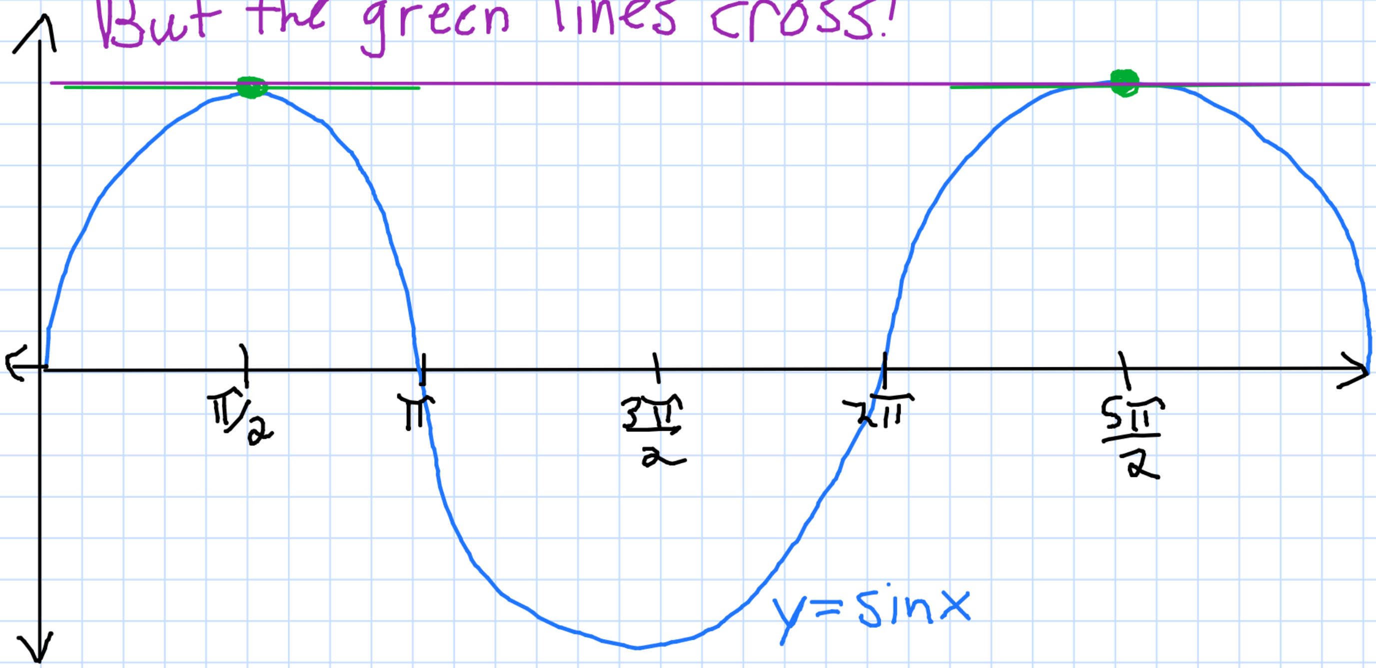


Graph of f(x) at a point P is a straight line that touches f(x) at point P, but does not cross f(x)

Note that this def doesn't work all the time



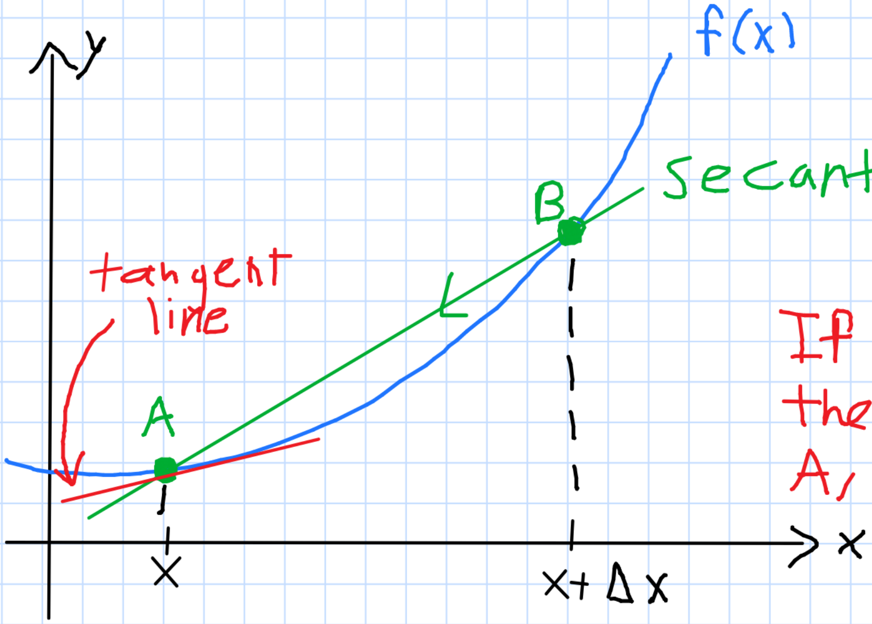
But the green lines cross!



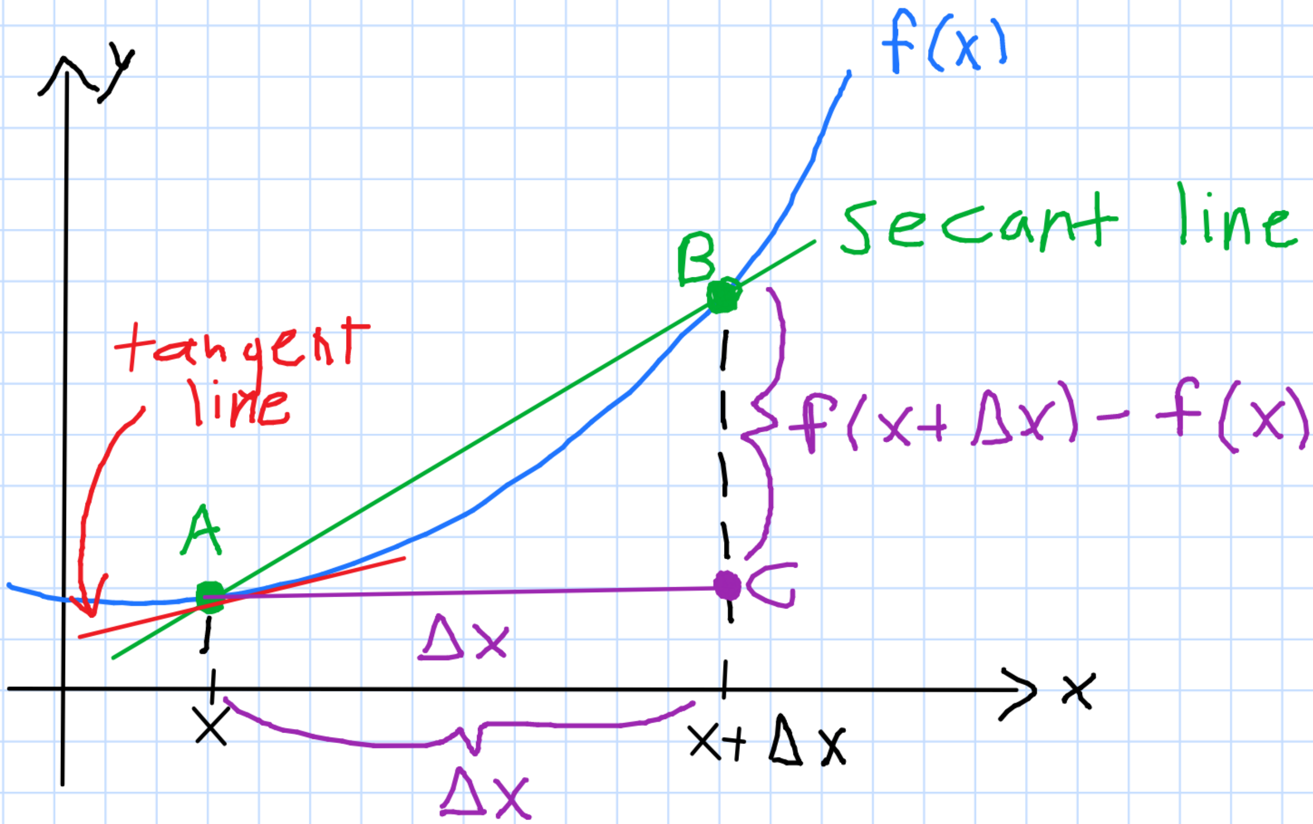
We need a precise definition for all scenarios.

Before we do so, let's recap secant lines.

A secant line of f(x) is a straight line that goes through 2 distinct pts on f(x).



If we move B along the curve f(x) toward A, then the secant line becomes tangent line

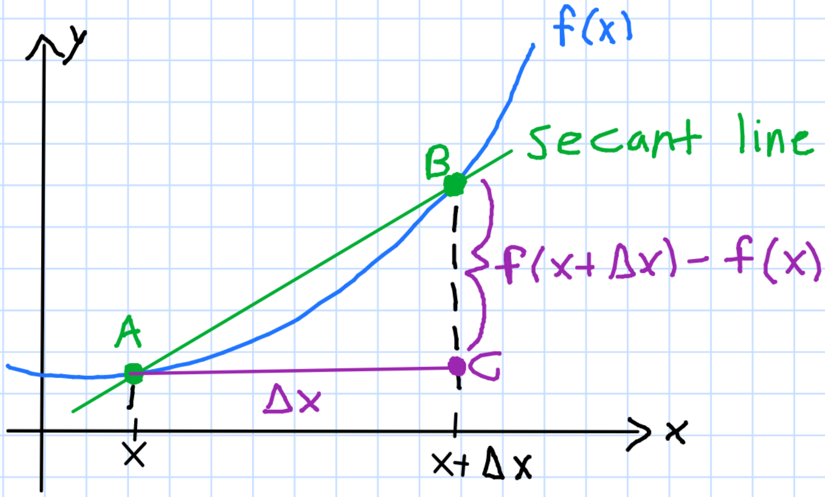


If we want to find tangent line to f(x) at a specific pt, then how can we achieve that?

If we knew slope of the tangent line, then we can use pt-slope formula to find eqn of tangent line.

Recall pt-slope formula:
Given m-slope and (x_1, y_1)-point,

$$y - y_1 = m(x - x_1)$$



Also know as difference quotient

$$\text{Slope of tangent line} = \lim_{\Delta x \to 0} \text{slope of secant line} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

How is the slope of the tangent line related to the derivative? They are the same!!

Definition: The derivative of $f(x)$ at x, denoted by $f'(x)$, is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \quad \text{where } h = \Delta x$$

provided that the limit exists.

Different Notations: y', $\frac{dy}{dx}$, $f'(x)$, $\frac{d}{dx}[f(x)]$

Game Plan: To find derivative using limit definition, follow the following steps:

1. Find $f(x+h)$
2. Find $-f(x)$
3. Find $f(x+h)-f(x)$ by adding 1 + 2
4. Find $\frac{f(x+h)-f(x)}{h}$ by dividing 3 by h
5. Find $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ by taking $\lim_{h\to 0}$ of 4

Ex 1: Find the derivative of $f(x)=x+5$ using limit definition.

Step 1: $f(x+h)=(x+h)+5=\cancel{x}+h+\cancel{5}$

Step 2: $-f(x)=-(x+5)\quad=\cancel{-x}\quad\cancel{-5}$

Step 3: Add Step 1+2: $\quad h$

Step 4: Divide Step 3 by h: $\quad \frac{h}{h}=1$

Step 5: Take $\lim_{h\to 0}$ of Step 4: $\quad \lim_{h\to 0} 1 = 1 = f'(x)$

Useful Formulas

- $(a\pm b)^2=a^2\pm 2ab+b^2$
- $a^2-b^2=(a-b)(a+b)$

Ex 2: Given $f(x) = x^2 - 3$.

(a) Find the slope of the tangent line.

i.e. Find $f'(x)$.

Step 1: $f(x+h) = (x+h)^2 - 3 = \cancel{x^2} + 2xh + h^2 - \cancel{3}$

Step 2: $-f(x) = -(x^2 - 3) = \cancel{-x^2} \quad \cancel{+3}$

$$2xh + h^2$$

Step 3: Add Step 1+2:

Step 4: Divide Step 3 by h:

$$\frac{2xh + h^2}{h} = \frac{\cancel{h}(2x+h)}{\cancel{h}} = 2x + h$$

Step 5: Take $\lim_{h \to 0}$ of Step 4:

$$\lim_{h \to 0} (2x + h) = 2x + 0 = 2x = f'(x)$$

Game Plan: To find the eqn of the tangent line to $f(x)$ at the point $x = c$, follow the following steps.

1. Find $f'(x)$.
2. Calculate $f'(c)$
3. Calculate $f(c)$
4. Plug 2 + 3 into the point-slope formula

$$y - f(c) = f'(c)(x - c)$$

Ex 2: Given $f(x)=x^2-3$.

(b) Find the eqn of the tangent line to $f(x)$ at $x=2$.

Step 1: $f'(x)=2x$ (from part a)

Step 2: $f'(2)=2(2)=4$

Step 3: $f(2)=(2)^2-3=4-3=1$

Step 4: Point-Slope Formula

$$y-f(2)=f'(2)(x-2)$$

$$y-1=4(x-2)$$

$$y-1=4x-8$$

$$+1 \qquad +1$$

$$y=4x-7$$