Lesson 11: The Chain

The Chain Rule

Recall Composition of Functions.

$y = f(g(x))$

g - inner function

f - outer function

Ex: ⓐ $y = (3x+1)^2$

$f(x) = x^2 \qquad g(x) = 3x+1$

Ex: ⓑ $y = \sin^2 x = (\sin x)^2$

$f(x) = x^2 \qquad g(x) = \sin x$

ⓒ $y = \ln(2x)$

$f(x) = \ln x \qquad g(x) = 2x$

ⓓ $y = \sqrt[3]{2x+1} = (2x+1)^{1/3}$

$f(x) = x^{1/3} \qquad g(x) = 2x+1$

Ex 1: Find y' of $y=(3x+1)^2$.

Method 1: Expand

$$y=(3x)^2+2(3x)(1)+1^2$$
$$=9x^2+6x+1$$
$$y'=18x+6$$

Chain Rule

$$y=f(\underbrace{g(x)}_{u})$$

$$\frac{dy}{dx}=\frac{d}{dx}[f(u)]=\frac{df}{du}\cdot\frac{du}{dx}$$ (Format I)

$$\frac{d}{dx}[f(g(x))]=f'(g(x))\cdot g'(x)$$ (Format II)

Ex 1: $y=(3x+1)^2$

Method 2: Chain Rule

$f(x)=x^2$ $\qquad$ $g(x)=3x+1$

$f'(x)=2x$ $\qquad$ $g'(x)=3$

$$y' = f'(g(x)) \cdot g'(x)$$
$$= f'(3x+1) \cdot 3$$
$$= 2(3x+1) \cdot 3$$
$$= 6(3x+1)$$
$$= 18x+6$$

Ex 2: $y = 2\cos^3 x = 2(\cos x)^3$

$f(x) = 2x^3$ $\qquad$ $g(x) = \cos x$

$f'(x) = 6x^2$ $\qquad$ $g'(x) = -\sin x$

$$y' = f'(g(x))g'(x)$$
$$= f'(\cos x)(-\sin x)$$
$$= 6(\cos x)^2(-\sin x)$$
$$= -6\cos^2 x \sin x$$

Ex 3: $y = \left(\dfrac{2x}{3x^2+x}\right)^3$

$f(x) = x^3$ $\qquad$ $g(x) = \dfrac{2x}{3x^2+x} = \dfrac{2x}{x(3x+1)} = \dfrac{2}{3x+1}$

$f'(x) = 3x^2$ $\qquad$ $u(x) = 2 \quad v(x) = 3x+1$

$u'(x) = 0 \quad v'(x) = 3$

$$g'(x) = \frac{u'v - v'u}{v^2}$$
$$= \frac{0\cdot(3x+1) - 3\cdot 2}{(3x+1)^2} = \frac{-6}{(3x+1)^2}$$

Ex 3: $y = \left(\frac{2x}{3x^2+x}\right)^3$

$f(x) = x^3 \quad g(x) = \frac{2}{3x+1}$

$f'(x) = 3x^2 \quad g'(x) = \frac{-6}{(3x+1)^2}$

$$y' = f'(g(x))\cdot g'(x) = f'\left(\frac{2}{3x+1}\right)\left(\frac{-6}{(3x+1)^2}\right)$$
$$= 3\left(\frac{2}{3x+1}\right)^2\left(\frac{-6}{(3x+1)^2}\right)$$

$$= 3 \cdot \frac{4}{(3x+1)^2} \cdot \frac{-6}{(3x+1)^2} = \frac{-72}{(3x+1)^4}$$

Ex 4: $y = \frac{15}{\sqrt[3]{x^2+1}} = \frac{15}{(x^2+1)^{1/3}} = 15(x^2+1)^{-1/3}$

$f(x) = 15x^{-1/3}$ $\quad g(x) = x^2+1$

$f'(x) = 15\left(-\frac{1}{3}\right)x^{-4/3}$ $\quad g'(x) = 2x$

$= -5x^{-4/3}$

$$y' = f'(g(x))\, g'(x) = f'(x^2+1)(2x) = -5(x^2+1)^{-4/3}(2x)$$

$$= -5 \cdot \frac{1}{(x^2+1)^{4/3}} \cdot 2x = \frac{-10x}{(x^2+1)^{4/3}}$$

Trigonometric & Exponential Functions w/ Chain Rule

- $\sin^2 x = (\sin x)^2$
- $\sin(x^2)$ $\neq$

$x = \pi/2$

$\sin^2 x = (\sin(\pi/2))^2 = 1^2$

$\sin(x^2) = \sin((\pi/2)^2)$ $\neq$

$= \sin(\pi^2/4)$

Ex 5: $y = \sin(x^2)$