Lesson 12: The Chain Rule; Derivative of the Natural Logarithmic Function

Lesson 12: More Chain Rule

Recall $y = f(g(x))$

$y' = f'(g(x))\, g'(x)$

Ex 6: $y = \sec(-2x+1)\tan(3x)$. Find y'.

$u(x) = \sec(-2x+1)$

$f(x) = \sec x$ $\qquad g(x) = -2x+1$

$f'(x) = \sec x \tan x$ $\qquad g'(x) = -2$

$u'(x) = f'(g(x))\, g'(x)$

$= f'(-2x+1)(-2)$

$= -2\sec(-2x+1)\tan(-2x+1)$

$v(x) = \tan(3x)$

$f(x) = \tan x$ $\qquad g(x) = 3x$

$f'(x) = \sec^2 x$ $\qquad g'(x) = 3$

$v'(x) = f'(g(x))\, g'(x)$

$= f'(3x)(3) = 3\sec^2(3x)$

Ex 6: $y = \sec(-2x+1)\tan(3x)$. Find y'.

$u(x) = \sec(-2x+1)$ $\quad$ $v(x) = \tan(3x)$

$u'(x) = -2\sec(-2x+1)\tan(-2x+1)$ $\quad$ $v'(x) = 3\sec^2(3x)$

$$y' = u'v + v'u$$
$$= -2\sec(-2x+1)\tan(-2x+1)\tan(3x) + 3\sec^2(3x)\sec(-2x+1)$$

Ex 7: $y = e^{(1-2x)^4} = \exp[(1-2x)^4]$

$f(x) = e^x$

$f'(x) = e^x$

$g(x) = (1-2x)^4$

$a(x) = x^4$ $\quad$ $b(x) = 1-2x$

$a'(x) = 4x^3$ $\quad$ $b'(x) = -2$

$$g'(x) = a'(b(x))\,b'(x)$$
$$= a'(1-2x)(-2)$$
$$= -2(4)(1-2x)^3$$
$$= -8(1-2x)^3$$

Ex 7: $y = e^{(1-2x)^4} = \exp[(1-2x)^4]$

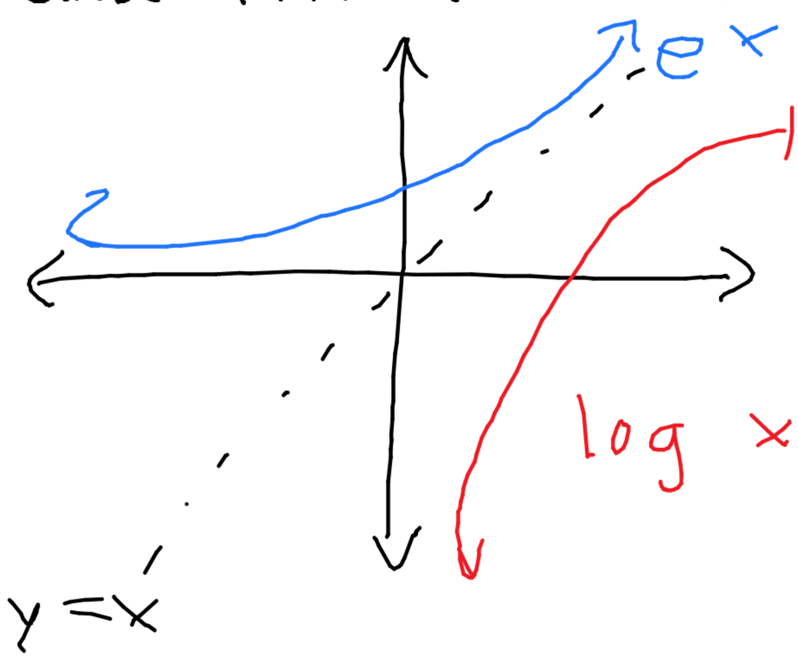
$f(x) = e^x$ $\quad g(x) = (1-2x)^4$

$f'(x) = e^x$ $\quad g'(x) = -8(1-2x)^3$

$$y' = f'(g(x))\,g'(x)$$
$$= f'((1-2x)^4)(-8)(1-2x)^3$$
$$= -8(1-2x)^3 \exp[(1-2x)^4]$$

Derivative of Logarithmic Functions

Recall e^x and $\ln x$ are inverses.



At $(1,e)$, what is the slope of e^x?

$$m = \frac{d}{dx}\bigg|_{(1,e)} (e^x) = e^x\bigg|_{(1,e)} = e$$

Since e^x and $\ln x$ are inverses,

$$\text{Slope of } e^x = \frac{1}{\text{Slope of } \ln x}$$

In others, slope of $\ln x$ is the reciprocal of the first coordinate pair.

Derivative of $\ln x$: $x > 0$

$$\frac{d}{dx}[\ln x] = \frac{1}{x}$$

Ex 1: $y = x\ln x$. Find y'.

$u(x) = x \qquad v(x) = \ln x$

$u'(x) = 1 \qquad v'(x) = \frac{1}{x}$

$$y' = u'v + v'u = 1 \cdot \ln x + x \cdot \frac{1}{x} = \ln x + 1$$

Derivative of $\ln[u(x)]$

$$y = \ln[u(x)]$$

$$f(x) = \ln x \qquad g(x) = u(x)$$

$$f'(x) = \frac{1}{x} \qquad g'(x) = u'(x)$$

$$y' = f'(g(x))\, g'(x)$$

$$= f'(u(x))\, u'(x)$$

$$= \frac{1}{u(x)} \cdot u'(x) = \frac{u'(x)}{u(x)}$$

Ex 3: $y = \ln \sqrt[3]{\dfrac{x^2+1}{2x-1}}$. Find y'.

$$y = \ln\left(\frac{x^2+1}{2x-1}\right)^{1/3}$$

Method 1: Start off w/ Chain Rule

$f(x) = \ln x$ $\qquad g(x) = \left(\dfrac{x^2+1}{2x-1}\right)^{1/3}$

$a(x) = x^{1/3}$ $\qquad b(x) = \dfrac{x^2+1}{2x-1}$

$u(x) = x^2+1$ $\qquad v(x) = 2x-1$

$u'(x) = 2x$ $\qquad v'(x) = 2$

$a'(x) = \frac{1}{3}x^{-2/3}$ $\qquad b'(x) = \dfrac{2x(2x-1)-2(x^2+1)}{(2x-1)^2}$

$$= \frac{4x^2-2x-2x^2-2}{(2x-1)^2}$$

$$= \frac{2x^2-2x-2}{(2x-1)^2}$$

$f'(x) = \dfrac{1}{x}$

$$g'(x) = a'(b(x))\,b'(x)$$

$$= a'\left(\frac{x^2+1}{2x-1}\right)\left(\frac{2x^2-2x-2}{(2x-1)^2}\right)$$

$$= \frac{1}{3}\left(\frac{x^2+1}{2x-1}\right)^{-2/3}\left(\frac{2x^2-2x-2}{(2x-1)^2}\right)$$

$$y' = f'(g(x))\,g'(x)$$

$$= f'\left[\left(\frac{x^2+1}{2x-1}\right)^{1/3}\right]\frac{1}{3}\left(\frac{x^2+1}{2x-1}\right)^{-2/3}\left(\frac{2x^2-2x-2}{(2x-1)^2}\right)$$

$$= \frac{1}{\left(\frac{x^2+1}{2x-1}\right)^{1/3}} \cdot \frac{1}{3} \cdot \left(\frac{x^2+1}{2x-1}\right)^{-2/3} \cdot \frac{2(x^2-x-1)}{(2x-1)^2}$$

$$= \left(\frac{x^2+1}{2x-1}\right)^{-1/3} \cdot \frac{1}{3} \cdot \left(\frac{x^2+1}{2x-1}\right)^{-2/3} \cdot \frac{2(x^2-x-1)}{(2x-1)^2}$$

$$= \frac{(x^2+1)^{\boxed{-1/3}}}{(2x-1)^{\boxed{-1/3}}} \cdot \frac{1}{3} \cdot \frac{(x^2+1)^{\boxed{-2/3}}}{(2x-1)^{\boxed{-2/3}}} \cdot \frac{2(x^2-x-1)}{(2x-1)^{\boxed{2}}}$$

$$= \frac{2}{3} \cdot \frac{(x^2-x-1)(x^2+1)^{-1/3-2/3}}{(2x-1)^{-1/3-2/3+2}}$$

$$= \frac{2}{3} \cdot \frac{(x^2-x-1)(x^2+1)^{-1}}{(2x-1)}$$

$$= \frac{2}{3} \cdot \frac{(x^2-x-1)}{(x^2+1)(2x-1)}$$

Method 2: Expand out w/ log properties

$$y = \ln\left(\frac{x^2+1}{2x-1}\right)^{1/3} = \frac{1}{3}\ln\left(\frac{x^2+1}{2x-1}\right) = \frac{1}{3}\left[\underbrace{\ln(x^2+1)} - \underbrace{\ln(2x-1)}\right]$$

$$y' = \frac{1}{3}\left[\frac{1}{x^2+1}\cdot 2x - \frac{1}{2x-1}\cdot 2\right] = \frac{2}{3}\left[\frac{x}{x^2+1} - \frac{1}{2x-1}\right]$$

$$= \frac{2}{3}\left[\frac{x(2x-1)-(x^2+1)}{(x^2+1)(2x-1)}\right]$$

$$= \frac{2}{3}\left[\frac{2x^2-x-x^2-1}{(x^2+1)(2x-1)}\right]$$

$$= \frac{2}{3} \cdot \frac{x^2-x-1}{(x^2+1)(2x-1)}$$