# Lesson 14: Implicit Differentiation

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Explicit Form: $y = f(x)$

Implicit Form: When a function is not written in explicit form.

Ex: (1) $y - 2x = 1$

(2) $x^2 + y^2 = 2$

(3) $y^2 + y - 1 = x$

To differentiate functions of this kind, we use a technique called implicit differentiation.

Here is where the notation $\frac{d}{dx}$ will come to play.

Ex 1: Use implicit differentiation to find slope of tangent line of

$$x^2 - y^2 = 4x + 8y \text{ @ } (0,0).$$

$$\frac{d}{dx}(x^2 - y^2) = \frac{d}{dx}(4x + 8y)$$

$$\frac{d}{dx}(x^2) - \frac{d}{dx}(y^2) = \frac{d}{dx}(4x) + \frac{d}{dx}(8y)$$

$$2x \cancel{\frac{dx}{dx}}^{1} - 2y\frac{dy}{dx} = 4\cancel{\frac{dx}{dx}}^{1} + 8\frac{dy}{dx}$$

$$2x - 2y\frac{dy}{dx} = 4 + 8\frac{dy}{dx}$$

Solve for dy/dx.

$$2x - 4 = 8\frac{dy}{dx} + 2y\frac{dy}{dx}$$

$$2x - 4 = (8 + 2y)\frac{dy}{dx}$$

$$\frac{2x-4}{8+2y} = \frac{dy}{dx} \qquad \left.\frac{dy}{dx}\right|_{(0,0)} = \frac{-4}{8} = -\frac{1}{2}$$

Ex 2: Find the slope of tangent lines to $x^2+y^2=4$ @ $(1,\sqrt{3})$, $(1,-\sqrt{3})$

$$\frac{d}{dx}(x^2+y^2)=\frac{d}{dx}(4)$$

$$\frac{d}{dx}(x^2)+\frac{d}{dx}(y^2)=\frac{d}{dx}(4)$$

$$2x\underbrace{\cancel{\frac{dx}{dx}}}_{1}+2y\frac{dy}{dx}=0$$

$$2x+2y\frac{dy}{dx}=0$$

$$2y\frac{dy}{dx}=-2x$$

$$\frac{dy}{dx}=\frac{-2x}{2y}=\frac{-x}{y}$$

$(1,\sqrt{3})$: $\left.\frac{dy}{dx}\right|_{(1,\sqrt{3})}=\frac{-1}{\sqrt{3}}$

$(1,-\sqrt{3})$: $\left.\frac{dy}{dx}\right|_{(1,-\sqrt{3})}=\frac{-1}{-\sqrt{3}}=\frac{1}{\sqrt{3}}$

Ex 3: Find $dy/dx$ for

$$yx^2 + e^y = x$$

$$\frac{d}{dx}(yx^2 + e^y) = \frac{d}{dx}(x)$$

$$\underbrace{\frac{d}{dx}(yx^2)} + \frac{d}{dx}(e^y) = \frac{d}{dx}(x)$$

$$\frac{d}{dx}(y)\,x^2 + y\frac{d}{dx}(x^2) + \frac{d}{dx}(e^y) = \frac{d}{dx}(x)$$

$$1\frac{dy}{dx}x^2 + y \cdot 2x\,\cancel{\frac{dx}{dx}} + e^y\frac{dy}{dx} = 1 \cdot \cancel{\frac{dx}{dx}}$$

$$x^2\frac{dy}{dx} + 2xy + e^y\frac{dy}{dx} = 1$$

$$x^2\frac{dy}{dx} + e^y\frac{dy}{dx} = 1 - 2xy$$

$$(x^2 + e^y)\frac{dy}{dx} = 1 - 2xy \rightarrow \boxed{\frac{dy}{dx} = \frac{1-2xy}{x^2+e^y}}$$

Extra credit 2

$$\frac{d}{dx}(xy) = \frac{d}{dx}(x)\, y + x\frac{d}{dx}(y)$$
$$= 1\frac{dx}{dx}y + x \cdot 1\frac{dy}{dx}$$
$$= y + x\frac{dy}{dx}$$

$\frac{d}{dx}\left(\frac{x}{y}\right)$ using quotient rule

Ex 3: Find $y' = dy/dx$ for
$$4\sin x \cdot \cos y = 3$$

$$4\frac{d}{dx}(\sin x \cos y) = \frac{d}{dx}(3)$$
$$4\left[\frac{d}{dx}(\sin x)\cos y + \sin x\frac{d}{dx}(\cos y)\right] = \frac{d}{dx}(3)$$

$$4\left[\cos x \underbrace{\cancel{\frac{dx}{dx}}}_{1} \cos y + \sin x(-\sin y)\frac{dy}{dx}\right] = 0$$
$$\cos x \cos y - \sin x \sin y \frac{dy}{dx} = 0$$

$$\cos x \cos y = \sin x \sin y \frac{dy}{dx}$$

$$\frac{\cos x \cos y}{\sin x \sin y} = \frac{dy}{dx} = \cot x \cot y$$

**Ex 5:** Find $dy/dx$ for

$$ye^x = 2\ln y$$

$$\frac{d}{dx}(ye^x) = 2\frac{d}{dx}(\ln y)$$

$$\frac{d}{dx}(y)e^x + y\frac{d}{dx}(e^x) = 2\frac{d}{dx}(\ln y)$$

$$1\cdot\frac{dy}{dx}e^x + ye^x\underbrace{\cancel{\frac{dx}{dx}}}_{1} = 2\cdot\frac{1}{y}\cdot\frac{dy}{dx}$$

$$e^x\frac{dy}{dx} + ye^x = \frac{2}{y}\frac{dy}{dx}$$

$$ye^x = \frac{2}{y}\frac{dy}{dx} - e^x\frac{dy}{dx}$$

$$ye^x = \left(\frac{2}{y} - e^x\right)\frac{dy}{dx} = \left(\frac{2 - e^x y}{y}\right)\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{ye^x}{\frac{2}{y} - e^x} = \frac{dy}{dx} = \frac{ye^x}{1}\cdot\frac{y}{2 - e^x y}$$

$$= \frac{y^2 e^x}{2 - e^x y}$$

$y = \ln x \quad \frac{dy}{dx} = \frac{1}{x}$

Proof: $y = \ln x \iff e^y = x$

$\frac{d}{dx}(e^y) = \frac{d}{dx}(x)$

$e^y \frac{dy}{dx} = 1 \quad \cancel{\frac{dx}{dx}} \Rightarrow 1$

$e^y \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{e^y}$

$\frac{dy}{dx} = \frac{1}{x}$

Q.E.D

$y = \ln x \quad \frac{dy}{dx} = \frac{1}{x}$

Proof: $y = \ln x \iff e^y = x$