LESSONS 15-16: RELATED RATES

This space is left for you to take your own notes.

Related Rates are word problems that
use Implicit differentiation.
We will be taking the derivative of
equations with respect to time, t.

Recipe for Solving a Related Rates Problem

Step 1: Draw a good picture. Label all constant values and give variable names to any changing quantities.

Step 2: Determine what information you **KNOW** and what you **WANT** to find.

Step 3: Find an equation relating the relevant variables. This usually involves a formula from geometry, similar triangles, the Pythagorean Theorem, or a formula from trigonometry. Use your picture!

Step 4: Use implicit differentiation to differentiate the equation with respect to time t.

Step 5: Substitute in what you **KNOW** from **Step 2** and any information that your equation in **Step 3** can give you and solve for the quantity you **WANT**. Do **NOT** substitute before this step!

Some Useful Formulas

Right Triangle
Pythagorean Theorem:
$$a^2 + b^2 = c^2$$

Triangle
$A = \frac{1}{2} bh$
$P = a + b + c$

Equilateral Triangle
$h = \frac{\sqrt{3}}{2}s \qquad A = \frac{\sqrt{3}}{4}s^2$

Rectangle
$A = lw$
$P = 2l + 2w$

Trapezoid
$A = \frac{1}{2}(a + b)h$

Circle
$A = \pi r^2 \qquad C = 2\pi r$

Circular Sector
$A = \frac{1}{2} r^2\theta \qquad s = r\theta$

Circular Ring
$A = \pi(R^2 - r^2)$

Rectangular Box
$V = lwh$
$S = 2(hl + lw + hw)$

Sphere
$V = \frac{4}{3}\pi r^3 \qquad S = 4\pi r^2$

Right Circular Cylinder
$V = \pi r^2 h \qquad S = 2\pi rh$

Right Circular Cone
$V = \frac{1}{3}\pi r^2 h$

Example 1: If x and y are both functions of t and $x + y^3 = 2$,

(1) Find $\frac{dy}{dt}$ when $\frac{dx}{dt} = -2$ and $y = 1$.

$$\frac{d}{dt}(x + y^3) = \frac{d}{dt}(2)$$

$$\frac{d}{dt}(x) + \frac{d}{dt}(y^3) = \frac{d}{dt}(2)$$

$$1 \cdot \frac{dx}{dt} + 3y^2 \frac{dy}{dt} = 0$$

$$-2 + 3(1)^2 \frac{dy}{dt} = 0$$

$$\frac{dy}{dt} = \frac{2}{3}$$

(2) Find $\frac{dx}{dt}$ when $\frac{dy}{dt} = 3$ and $x = 1$.

$$\frac{dx}{dt} + 3y^2(3) = 0$$

$$\frac{dx}{dt} = -9y^2$$

$$= -9(1)^2$$

$$= -9$$

$$x + y^3 = 2$$
$$1 + y^3 = 2$$
$$y^3 = 1$$
$$y = 1$$

Example 2: A Spherical balloon is being deflated at a constant rate of 20 cubic cm per second. How fast is the radius of the balloon changing at the instant when the balloon's radius is 12 cm? 16 cm? Is the radius changing more rapidly when r = 12 cm or when r = 16 cm? Does it make sense?

Step 1: Draw a picture. Label all constant values and give variable names to any changing quantities.
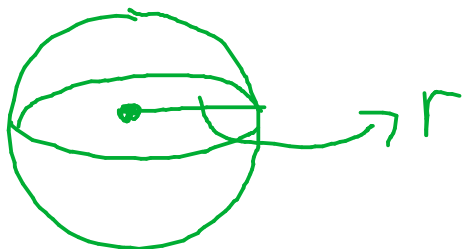
Step 2: Determine what information you **KNOW** and what you **WANT** to find.

KNOW: $\dfrac{dV}{dt} = -20 \dfrac{cm^3}{s}$ **WANT:** $\dfrac{dr}{dt}\Big|_{r=12}$

$\dfrac{dr}{dt}\Big|_{r=16}$

Step 3: Find an equation relating the relevant variables.

$$V = \dfrac{4}{3}\pi r^3$$

Step 4: Use implicit differentiation to differentiate the equation with respect to time t.

$$\dfrac{d}{dt}(V) = \dfrac{d}{dt}\left(\dfrac{4}{3}\pi r^3\right)$$

$$\dfrac{dV}{dt} = \dfrac{4}{3}\pi \, 3r^2 \, \dfrac{dr}{dt}$$

$$= 4\pi r^2 \dfrac{dr}{dt}$$

Step 5: Substitute in what you **KNOW** from **Step 2** and any information that your equation in **Step 3** can give you and solve for the quantity you **WANT**.

$\boxed{r=12}$

$-20 = 4\pi(12)^2 \dfrac{dr}{dt}$

$\dfrac{dr}{dt} = \dfrac{-20}{4\pi(12)^2} = \dfrac{-5}{144\pi}$

$\boxed{r=16}$

$-20 = 4\pi(16)^2 \dfrac{dr}{dt}$

$\dfrac{dr}{dt} = \dfrac{-20}{4\pi(16)^2} = \dfrac{-5}{256\pi}$

Example 3: A ladder 5 meters long rests on horizontal ground and leans against a vertical wall. The foot of the ladder is pulled away from the wall at the rate of 0.3 m/sec. How fast is the top sliding down the wall when the foot of the ladder is 3m from the wall?

Step 1: Draw a picture. Label all constant values and give variable names to any changing quantities.

$$5$$

$$x$$

$$y$$
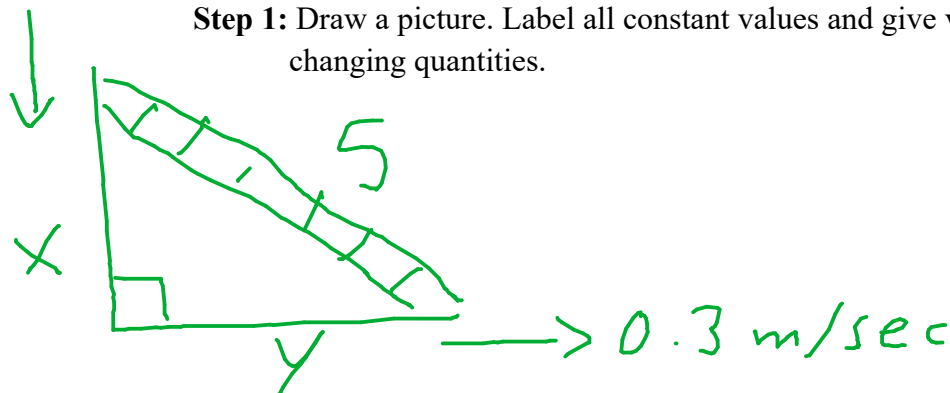
$$\rightarrow 0.3 \, m/sec$$

Step 2: Determine what information you **KNOW** and what you **WANT** to find.

KNOW: $\dfrac{dy}{dt} = 0.3 \dfrac{m}{sec}$

WANT: $\dfrac{dx}{dt}\Big|_{y=3}$

Step 3: Find an equation relating the relevant variables.

$$x^2 + y^2 = 5^2 = 25$$

Step 4: Use implicit differentiation to differentiate the equation with respect to time t.

$$2x \dfrac{dx}{dt} + 2y \dfrac{dy}{dt} = 0$$

$$x \dfrac{dx}{dt} + y \dfrac{dy}{dt} = 0$$

Step 5: Substitute in what you **KNOW** from **Step 2** and any information that your equation in **Step 3** can give you and solve for the quantity you **WANT**.

$$x \dfrac{dx}{dt} + 3(0.3) = 0$$

$$\dfrac{dx}{dt} = \dfrac{-0.9}{x} = \dfrac{-0.9}{4} = -0.225$$

$$x^2 + y^2 = 25$$
$$x^2 + 3^2 = 25$$
$$x^2 = 16$$
$$x = 4$$

Example 4: A plane is flying directly away from you at 500 mph at an altitude of 3 miles.

(1) How fast is the plane's distance from you increasing at the moment when the plane is flying over a point on the ground 4 miles from you?

Step 1: Draw a picture. Label all constant values and give variable names to any changing quantities.

$500mph \rightarrow$

3

z

3

x
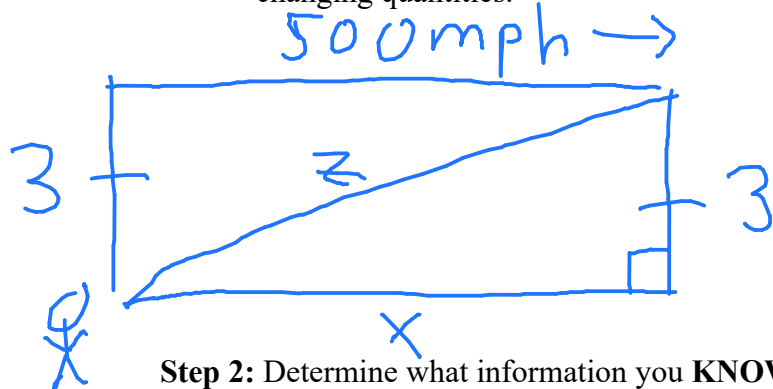
Step 2: Determine what information you **KNOW** and what you **WANT** to find.

KNOW: $\dfrac{dx}{dt} = 500 \, mph$ **WANT:** $\dfrac{dz}{dt}\Big|_{x=4}$

Step 3: Find an equation relating the relevant variables.

$$x^2 + 3^2 = z^2 \iff x^2 + 9 = z^2$$

Step 4: Use implicit differentiation to differentiate the equation with respect to time t.

$$2x\dfrac{dx}{dt} = 2z\dfrac{dz}{dt}$$

$$x\dfrac{dx}{dt} = z\dfrac{dz}{dt}$$

Step 5: Substitute in what you **KNOW** from **Step 2** and any information that your equation in **Step 3** can give you and solve for the quantity you **WANT**.
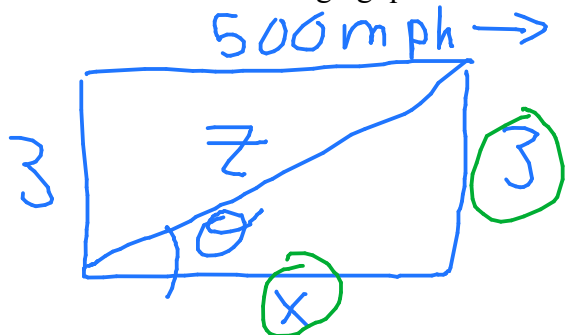
$$4(500) = z\dfrac{dz}{dt}$$

$$\dfrac{dz}{dt} = \dfrac{2000}{z} = \dfrac{2000}{5} = \boxed{400}$$

$x^2 + 9 = z^2$
$4^2 + 9 = z^2$
$25 = z^2$
$z = 5$

Example 4: A plane is flying directly away from you at 500 mph at an altitude of 3 miles.
    (2) How fast is the angle of elevation changing when it is $\pi/3$?

        Step 1: Draw a picture. Label all constant values and give variable names to any
            changing quantities.

$$500 \, mph \rightarrow$$

$$3 \qquad z \qquad \boxed{3}$$

$$\theta$$

$$\boxed{x}$$

        Step 2: Determine what information you **KNOW** and what you **WANT** to find.

         KNOW: $\dfrac{dx}{dt} = 500$ **WANT:** $\dfrac{d\theta}{dt}\Big|_{\theta = \pi/3}$

        Step 3: Find an equation relating the relevant variables.

$$\tan\theta = \frac{3}{x} = 3x^{-1}$$

        Step 4: Use implicit differentiation to differentiate the equation with respect to time t.

$$\sec^2\theta \, \frac{d\theta}{dt} = -3x^{-2}\frac{dx}{dt} = \frac{-3}{x^2}\frac{dx}{dt}$$

$$\sec^2(\pi/3)\frac{d\theta}{dt} = -\frac{3}{x^2}(500)$$

$$4 \, \frac{d\theta}{dt} = -\frac{1500}{x^2} \quad \Longleftrightarrow \quad \frac{d\theta}{dt} = \frac{-75}{x^2} = \frac{-75}{3}$$

        Step 5: Substitute in what you **KNOW** from **Step 2** and any information that your
            equation in **Step 3** can give you and solve for the quantity you **WANT**.

$$x^2 = 9/3$$
$$= 3$$
$$= -25$$

$$\tan\theta = \frac{3}{x} \quad \Bigg| \quad \frac{\sqrt{3}}{1} = \frac{3}{x} \quad \Bigg| \quad$$

$$\tan\left(\frac{\pi}{3}\right) = \frac{3}{x} \quad \Bigg| \quad \sqrt{3}\,x = 3$$

$$x = 3/\sqrt{3}$$

HW 15.4: A cylindrical tank standing upright (with one circular base on the ground) has a radius of 22 cm for the base. How fast does the water level in the tank drop when the water is being drained at 28 cm^3/sec? Note: The formula right circular cylinder is $V = \pi r^2 h$.

       Step 1: Draw a picture. Label all constant values and give variable names to any changing quantities.

       Step 2: Determine what information you **KNOW** and what you **WANT** to find.

           KNOW: **WANT:**

       Step 3: Find an equation relating the relevant variables.

       Step 4: Use implicit differentiation to differentiate the equation with respect to time t.

       Step 5: Substitute in what you **KNOW** from **Step 2** and any information that your equation in **Step 3** can give you and solve for the quantity you **WANT**.

HW 16.3: A baseball diamond is a square 90 ft on a side. A player runs from first base to second base at 14 ft/sec. At what rate is the player's distance from home base increasing when he is half way from first to second base?

Step 1: Draw a picture. Label all constant values and give variable names to any changing quantities.

Step 2: Determine what information you **KNOW** and what you **WANT** to find.

KNOW: **WANT:**

Step 3: Find an equation relating the relevant variables.

Step 4: Use implicit differentiation to differentiate the equation with respect to time *t*.

Step 5: Substitute in what you **KNOW** from **Step 2** and any information that your equation in **Step 3** can give you and solve for the quantity you **WANT**.

HW 16.4: A boat is pulled in to a dock by a rope with one end attached to the front of the boat and the other ended passing through a ring attached to the dock at a point $\sqrt{85}$ ft higher than the front of the boat. The rope is being pulled through the ring at the rate of 0.1 ft/sec. How fast is the boat approaching the dock when 11 ft of rope are out?

        Step 1: Draw a picture. Label all constant values and give variable names to any changing quantities.

        Step 2: Determine what information you **KNOW** and what you **WANT** to find.

           KNOW: **WANT:**

        Step 3: Find an equation relating the relevant variables.

        Step 4: Use implicit differentiation to differentiate the equation with respect to time t.

        Step 5: Substitute in what you **KNOW** from **Step 2** and any information that your equation in **Step 3** can give you and solve for the quantity you **WANT**.