Lesson 19: Concavity, Inflection Points & The Second Derivative Test

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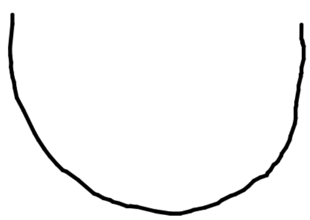
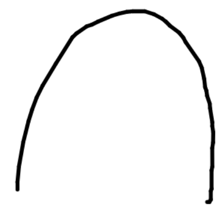
Concave Up



"like a cup"

Concave Down



"like a frown"

Concavity of a Function: suppose $f''(x)$ exists on an open interval I. Then

① If $\boxed{f''(x) > 0}$ for all x in I, then f(x) is $\boxed{\text{concave up}}$ on I.

② If $\boxed{f''(x) < 0}$ for all x in I, then f(x) is $\boxed{\text{concave down}}$ on I.

Ex1 Determine the largest open interval(s) on which graph of

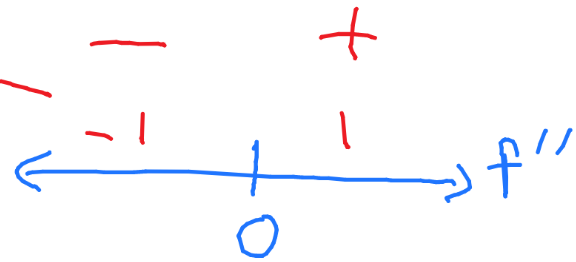
$$f(x) = x^3 - x$$

is concave up or down.

$f'(x) = 3x^2 - 1$

$f''(x) = 6x = 0$

$x = 0$



Concave Up: $(0, \infty)$ Concave Down: $(-\infty, 0)$

Ex2 Determine the largest open interval(s) on which graph of

$$f(x) = \frac{1}{12}x^4 - \frac{1}{3}x^3$$

is concave up or down.

$f'(x) = \frac{4}{12}x^3 - \frac{3}{3}x^2 = \frac{1}{3}x^3 - x^2$

$f''(x) = \frac{3}{3}x^2 - 2x = x^2 - 2x = 0$

$x(x-2) = 0$

$x = 0, x = 2$

$f''(-1) = -1(-1-2)$

$- \cdot - = +$

$f''(1) = 1(1-2)$

$+ \cdot - = -$

$\cup: (-\infty, 0) \cup (2, \infty)$ $\quad f''(3) = 3(3-2)$

$+ \cdot + = +$

$\cap: (0, 2)$

Steps in Finding the Inflection pts

① Find the pts on the curve where the second derivate is 0 or DNE.

i.e. Find x where $f''(x) = 0$ and $f''(x)$ DNE

② Test whether the concativity changes at these pts.

i.e.

or

where c is a pt found in ①

Ex 3: Find the inflection pt(s) of

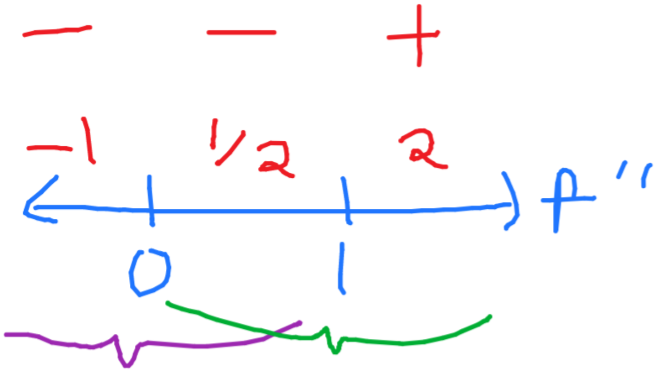
$f(x) = \frac{3}{5}x^5 - x^4$ if they exist.

$f'(x) = \frac{3}{\cancel{5}} \cdot \cancel{5}x^4 - 4x^3 = 3x^4 - 4x^3$

$f''(x) = 12x^3 - 12x^2 = 0$

$12x^2(x-1) = 0$

$x = 0, 1$



Is there any change? <u>No</u>

Is there any change? Yes

$\Rightarrow$ Inflection pt @ $x = 1$

$y = f(1)$

<u>Second Derivative Test</u>

Let $f(x)$ be a function such that $f'(c) = 0$ and $f''(x)$ exists on an open interval containing c.

① If $f''(c) > 0$, then $f(x)$ has a rel min $f(c)$ @ $x = c$.

② If $f''(c) < 0$, then $f(x)$ has a rel max $f(c)$ @ $x = c$.

If $f''(c) = 0$, the Second Derivative Test does not apply or fails.

It does not necessarily mean that we have neither a rel max or min at these pts.

It just means, if the Second Derivative Test fails just use the First Derivative Test.

Ex 4: Find all the rel extrema of $f(x) = x^3 - x$ if they exist.

(1) Find $f'(x) = 0$

$f'(x) = 3x^2 - 1 = 0$

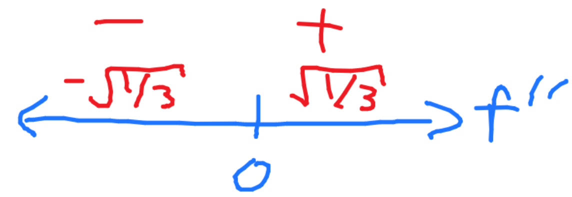
$x^2 = 1/3$

$x = \pm\sqrt{1/3}$

(2) Find $f''(x)$

$f''(x) = 6x$

(3) Plug (1) into (2).



$f''(-\sqrt{1/3}) < 0 \Rightarrow$ rel max @ $x = -\sqrt{1/3}$

$f''(\sqrt{1/3}) > 0 \Rightarrow$ rel min @ $x = \sqrt{1/3}$

Ex 5: Find all the rel extrema of $f(x) = \frac{3}{5}x^5 - x^4$ if they exist.

(1) Find $f'(x) = 0$

$f'(x) = 3x^4 - 4x^3 = 0$

$x^3(3x - 4) = 0$

$x = 0, 4/3$

(2) Find $f''(x)$

$f''(x) = 12x^3 - 12x^2 = 12x^2(x-1)$

③ Plug ① into ②.

$f''(4/3) = 12\left(\frac{4}{3}\right)^2\left(\frac{4}{3} - 1\right) > 0$

$\Rightarrow$ rel min @ $x = 4/3$

$f''(0) = 12(0)^2(0-1) = 0$

↳ Failed 2nd Derivative Test

↳ Use 1st Derivative Test

$f'(x) = 3x^4 - 4x^3 = x^3(3x-4)$

+ − +

−1 1 2

f'

0 4/3

By 1st derivative test, rel extrema. By case 2 or 1, $x=0$ rel max.

$f'(-1) = (-1)^3(-3-4)$

$- \cdot - = +$

$f'(1) = 1^3(3-4)$

$+ \cdot - = -$

$\cancel{f'(2) = 2^3(6-4)}$

$\cancel{+ \cdot + = +}$