Lesson 22: Limits at Infinity

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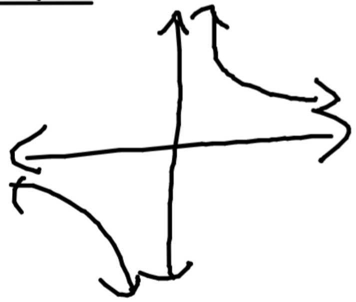
Recall $\lim\limits_{x\to\infty} \frac{1}{x} = 0$

$\lim\limits_{x\to-\infty} \frac{1}{x} = 0$



Purpose of Revisting this topic is to determine ① End Behavior
② If we have horizontal/slant asymptote

Ex 1: Find the following limits:

ⓐ $\lim\limits_{x\to\infty} \frac{3}{x} = 3 \lim\limits_{x\to\infty} \frac{1}{x} = 3\cdot 0 = 0$

ⓑ $\lim\limits_{x\to-\infty} \left(\frac{x}{3} + 2\right) = \frac{-\infty}{3} + 2 = -\infty$

General Rule: The limit of a rational function $f(x) = \frac{p(x)}{q(x)}$ as $x \to \pm\infty$ is determined by the leading terms of the numerator and the denominator.

A leading term of a polynomial is the term that has the highest power of x.

Ex 2: Find $\lim_{x\to\infty} \frac{2x^2+3}{x^2-1}$

$$\lim_{x\to\infty} \frac{2x^2+3}{x^2-1} = \text{"}\frac{\infty}{\infty}\text{"} \longrightarrow \text{No-no}$$

By General Rule,

$$\lim_{x\to\infty} \frac{2x^2}{x^2} = \lim_{x\to\infty} 2 = \textcircled{2}$$

Horizontal Asymptotes

The line $y=L$, where L is a constant, is a horizontal asymptote of $f(x)$ if

$$\lim_{x\to\infty} f(x) = L \quad \boxed{\text{or}} \quad \lim_{x\to-\infty} f(x) = L$$

$\downarrow$

Check both cases

Both limits need not match.

For ex. when $f(x)=e^x$

Ex 3: $h(x)=\frac{x-1}{x^2-1}$. Find the HA.

Method 1: Simplify h(x), first.

$$h(x)=\frac{\cancel{x-1}}{(\cancel{x-1})(x+1)}=\frac{1}{x+1}$$

$$\left.\begin{array}{l}\lim\limits_{x\to-\infty} h(x)=\lim\limits_{x\to-\infty}\frac{1}{x+1}=0\\ \lim\limits_{x\to\infty} h(x)=\lim\limits_{x\to\infty}\frac{1}{x+1}=0\end{array}\right\}\Rightarrow \text{HA } y=0$$

Method 2: Using the general rule,

$$\lim_{x\to-\infty} h(x)=\lim_{x\to-\infty}\frac{x}{x^2}=\lim_{x\to-\infty}\frac{1}{x}=0$$

$$\lim_{x\to\infty} h(x)=\lim_{x\to\infty}\frac{1}{x}=0 \longrightarrow \boxed{\text{HA } y=0}$$

Ex 2: $f(x)=\frac{x^3+5}{2x+1}$. Find the HA.

By general rule,

$$\lim_{x\to-\infty} f(x)=\lim_{x\to-\infty}\frac{x^3}{2x}=\lim_{x\to-\infty}\frac{x^2}{2}=\infty$$

$$\lim_{x\to\infty} f(x)=\lim_{x\to\infty}\frac{x^2}{2}=\infty$$

No HA b/c ∞ not a constant

Slant Asymptotes

The line $y=ax+b$, where a and b are constants and $a\neq 0$, is a slant asymptote of $f(x)$ if $f(x)$ gets closer and closer to $y=ax+b$ as $x\to\pm\infty$.

To find the slant asymptotes of a rational function, we use

- Synthetic Division
- Long Division

Synthetic Division Slide 1

Ex 5: Find the slant asymptotes of $y = \dfrac{x^2+2x+8}{x-2}$

To use Synthetic Division, find the zeros of the denominator of y.

i.e. $x-2=0 \Rightarrow x=2$

$$\begin{array}{c|ccc} 2 & & & \\ \hline & & & \end{array}$$

Synthetic Division Slide 2

Ex 5: Find the slant asymptotes of $y = \dfrac{x^2+2x+8}{x-2}$

To use Synthetic Division, find the zeros of the denominator of y.

i.e. $x-2=0 \Rightarrow x=2$

list the coefficients of the polynomial on the numerator

$$\begin{array}{c|ccc} 2 & 1 & 2 & 8 \\ & \downarrow & & \\ \hline & 1 & & \end{array}$$

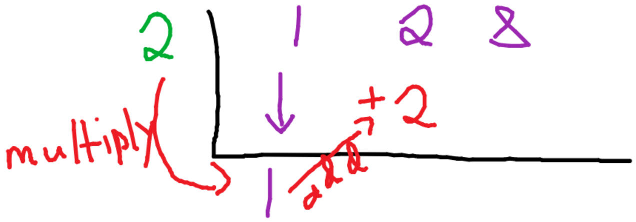
Synthetic Division Slide 3

Ex 5: Find the slant asymptotes of $y = \frac{x^2+2x+8}{x-2}$

To use Synthetic Division, find the zeros of the denominator of y.

i.e. $x-2=0 \Rightarrow x=2$

list the coefficients of the polynomial on the numerator



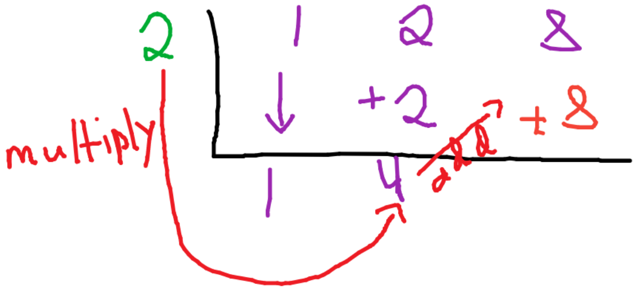
Synthetic Division Slide 4

Ex 5: Find the slant asymptotes of $y = \frac{x^2+2x+8}{x-2}$

To use Synthetic Division, find the zeros of the denominator of y.

i.e. $x-2=0 \Rightarrow x=2$

list the coefficients of the polynomial on the numerator



Synthetic Division Slide 5

Ex 5: Find the slant asymptotes of $y=\frac{x^2+2x+8}{x-2}$

To use Synthetic Division, find the zeros of the denominator of y.

i.e. $x-2=0 \Rightarrow x=2$

$$\begin{array}{r|rrr} 2 & 1 & 2 & 8 \\ & \downarrow & +2 & +8 \\ \hline & 1 & 4 & 16 \end{array}$$

↑remainder

$$y=\frac{x^2+2x+8}{x-2}$$

$$=x+4+\frac{16}{x-2}$$

Synthetic Division Slide 6

Ex 5: Find the slant asymptotes of $y=\frac{x^2+2x+8}{x-2}$

To use Synthetic Division, find the zeros of the denominator of y.

i.e. $x-2=0 \Rightarrow x=2$

$$\begin{array}{r|rrr} 2 & 1 & 2 & 8 \\ & \downarrow & +2 & +8 \\ \hline & 1 & 4 & 16 \end{array}$$

$$y=\frac{x^2+2x+8}{x-2}$$

$$=x+4+\frac{16}{x-2}$$

$$\boxed{\text{Slant: } y=x+4}$$

Long Division Slide 1

Ex 5: Find the slant asymptotes of $y=\frac{x^2+2x+8}{x-2}$

Long Division works w/o knowing the zeros of y's denominator. But involves alot of work.

$$\begin{array}{r} x \\ x-2\,\overline{\big)\,x^2+2x+8} \end{array}$$

① $\frac{x^2}{x}=x$

Long Division Slide 2

Ex 5: Find the slant asymptotes of $y=\frac{x^2+2x+8}{x-2}$

Long Division works w/o knowing the zeros of y's denominator. But involves alot of work.

$$\begin{array}{r} x \\ x-2\,\overline{\big)\,x^2+2x+8} \\ \underline{-(x^2-2x)} \end{array}$$

① $\frac{x^2}{x}=x$

② $x(x-2)=x^2-2x$

Long Division Slide 3

Ex 5: Find the slant asymptotes of $y = \frac{x^2+2x+8}{x-2}$

Long Division works w/o knowing the zeros of y's denominator. But involves alot of work.

$$\begin{array}{r|l} & \quad x \\ \hline x-2 & \cancel{x^2}+2x+8 \\ & -(\cancel{x^2}-2x)\ \downarrow \\ \hline & \qquad 4x+8 \end{array}$$

Repeat ① & ②.

Long Division Slide 4

Ex 5: Find the slant asymptotes of $y = \frac{x^2+2x+8}{x-2}$

Long Division works w/o knowing the zeros of y's denominator. But involves alot of work.

① $\frac{4x}{x} = 4$

$$\begin{array}{r|l} & \quad x+4 \\ \hline x-2 & x^2+2x+8 \\ & -(x^2-2x)\ \downarrow \\ \hline & \qquad 4x+8 \end{array}$$

Long Division Slide 5

Ex 5: Find the slant asymptotes of $y=\dfrac{x^2+2x+8}{x-2}$

Long Division works w/o knowing the zeros of y's denominator. But involves alot of work.

$$\begin{array}{r} x+4 \\ x-2\,\overline{)\,x^2+2x+8} \\ -\,x^2-2x)\ \downarrow \\ \hline 4x+8 \\ -(4x-8) \\ \hline \end{array}$$

① $\dfrac{4x}{x}=4$

② $4(x-2)=4x-8$

Long Division Slide 6

Ex 5: Find the slant asymptotes of $y=\dfrac{x^2+2x+8}{x-2}$

Long Division works w/o knowing the zeros of y's denominator. But involves alot of work.

$$\begin{array}{r} x+4 \\ x-2\,\overline{)\,x^2+2x+8} \\ -\,x^2-2x)\ \downarrow \\ \hline 4\cancel{x}+8 \\ -(\cancel{4}x-8) \\ \hline 16 \end{array}$$

$$y=\frac{x^2+2x+8}{x-2}$$
$$=x+4+\frac{16}{x-2}$$

$$\boxed{\underline{\text{Slant}}:\ y=x+4}$$

Lesson 22: Limits at Infinity

Ex 6: Find the slant asymptote of

$$g(x) = \frac{x^3 + x^2 + 1}{x + 3} = \frac{x^3 + x^2 + 0x + 1}{x + 3}$$

Sythetic Division

Zeros of $x+3 \Longrightarrow x = -3$

$$\begin{array}{r|rrrr} -3 & 1 & 1 & 0 & 1 \\ & \downarrow & -3 & 6 & -18 \\ \hline & 1 & -2 & 6 & -17 \end{array}$$

$$g(x) = \boxed{x^2 - 2x + 6} - \frac{17}{x+3}$$

$g(x)$ has no slant asym.

Recall a slant asymptote has the form

$$y = ax + b$$

So $g(x) = \frac{x^3 + x^2 + 1}{x + 3}$ doesn't have a slant asymptote.

You can avoid the work by checking the difference b/w the exponents of the leading terms of num. & deno. is equal to 1.