Lessons 24-26: Optimization

Optimization Problems: Often this includes finding the maximum or minimum value of some function.

- e.g. The minimum time to make a certain journey,
- The minimum cost for constructing some object,
- The maximum profit to gain for a business, and so on.

How do we solve an optimization problem?

- Determine a function (**known as objective function)** that we need to maximize or minimize.

- Determine if there are some constraints on the variables. (The equations that describe the constraints are called **the constraint equations**.)
  - If there are constraint equations, rewrite the **objective function** as a function of only one variable.

- Then we can solve for absolute maximum or minimum like we did before.
  ↳ via the 1st/2nd Derivative Test

This space is left for you to take your own notes.

Recipe for Solving an Optimization Problem

Step 1: Identify what quantity you are trying to optimize.

Step 2: Draw a picture (if applicable), corresponding to the problem, and label it with your variables.

Step 3: Express the variable to be optimized as a function of the variables you used in Step 2.

Step 4: Find relations among the variables from Step 2 and express the variable to be optimized a function of just one of the variables from Step 2.

Step 5: Identify the domain for the function you found in Step 4.

Step 6: Find the absolute maximum of the variable to be optimized on this domain.

min

Step 7: Reread the question and be sure you have answered exactly what was asked.

This space is left for you to take your own notes.

5 Be careful w/ ( ) or [ ]

e.g. No negatives for $$ or

lengths

Example 2: Of all the numbers whose sum is 50, find the two that have the maximum product.

Let x and y be such #s.

1. Product
2. None
3. $P = xy$
4. $x + y = 50$
5. Domain x and y:

   $(-\infty, \infty)$
6. Solve 4 for y.

   $y = 50 - x$

   Plug y into 3

   $P = x(50 - x)$

   $= 50x - x^2$

   Take the derivative set $= 0$.

   $P' = 50 - 2x = 0$

   $x = 25$

   Check that $x = 25$ gives abs max.
   Since you have no endpts, use 2nd Derivatives.

   $P'' = -2 < 0$

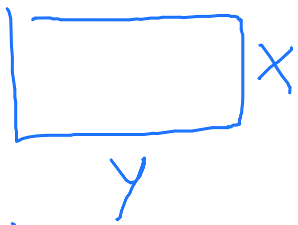
   $\Rightarrow$ abs max
7. $x = 25$

   $y = 50 - x = 25$

Example 3: A carpenter is building a rectangular room with a fixed perimeter of 54 feet. What are the dimensions of the largest room that can be built? What is its area?

1. Area

2.



3. $A = xy$

4. $54 = P = 2x + 2y$

$27 = x + y$

5. <u>Domain x and y</u>: $(0, 27)$

$x, y = 0 \Rightarrow$ no length

$27 = 0 + y$

$y = 27 \Rightarrow$ no width

6. Solve (4) for y.

$y = 27 - x$

Plug y into (3).

$A = x(27 - x)$

$= 27x - x^2$

Take derivative of A set $= 0$.

$A' = 27 - 2x = 0$

$x = 27/2$

$A'' = -2 < 0 \Rightarrow$ abs max

7. $x = 27/2$

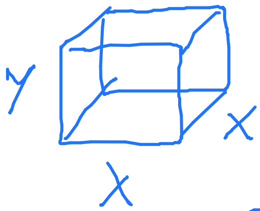
$y = 27/2$

$A = \dfrac{729}{4}$

Example 4: An open-top box with a square base is to have a volume of 8 cubic feet. Find the dimensions of the box that can be made with the least amount of material.

① Surface Area

②



③ $SA = x^2 + 4xy$

④ $8 = V = x^2 y$

⑤ Domain x and y:

$(0, \infty)$

⑥ Solve ④ for y.

$$y = \frac{8}{x^2}$$

Plug y into ③.

$$SA = x^2 + 4x\left(\frac{8}{x^2}\right)$$

$$= x^2 + \frac{32}{x}$$

$$= x^2 + 32x^{-1}$$

Take derivative of SA set $= 0$.

$$(SA)' = 2x - 32x^{-2} = 0$$

$$2x - \frac{32}{x^2} = 0$$

$$\frac{2x}{1} = \frac{32}{x^2}$$

$$2x^3 = 32$$

$$x^3 = 16$$

$$x = 2 \cdot \sqrt[3]{2}$$

Check abs min

$$(SA)'' = 2 - 32(-2)x^{-3}$$

$$= 2 + \frac{64}{x^3}$$

$$(SA)''(2 \cdot \sqrt[3]{2}) = 2 + \frac{64}{16} > 0$$

$\Rightarrow$ abs min

⑦ $x = 2 \cdot \sqrt[3]{2}$

$$y = \frac{8}{(2 \cdot \sqrt[3]{2})^2} = \sqrt[3]{2}$$

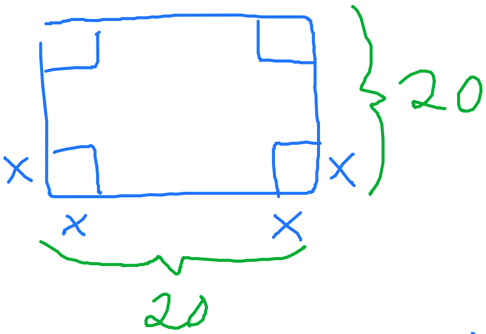
$2 \cdot \sqrt[3]{2} \times 2 \cdot \sqrt[3]{2} \times \sqrt[3]{2}$

ℓ $\quad$ w $\quad$ h

Example 1: From a thin piece of cardboard 20 in by 20 in, square corners are cut out so that the sides can be folded up to make a box. What dimensions will yield a box of maximum volume? What is the maximum volume?

① Volume

②



③ $V = x(20-2x)^2$

④ No constraints

⑤ Domain of x: $(0, 10)$

$x = 0 \Rightarrow$ no height

$20 - 2x = 0 \Rightarrow$ no width

$x = 10$

⑥ $V = x(400 - 80x + 4x^2)$

$= 400x - 80x^2 + 4x^3$

$V' = 400 - 160x + 12x^2 = 0$

$4(100 - 40x + 3x^2) = 0$

$4(10-3x)(10-x) = 0$

$x = \frac{10}{3}, \cancel{10}$ by ⑤

$V'' = -160 + 24x$

$V''(10/3) = -80 < 0 \Rightarrow$ abs max

⑦ $x = 10/3$

$20 - 2x = 40/3$

Dimensions

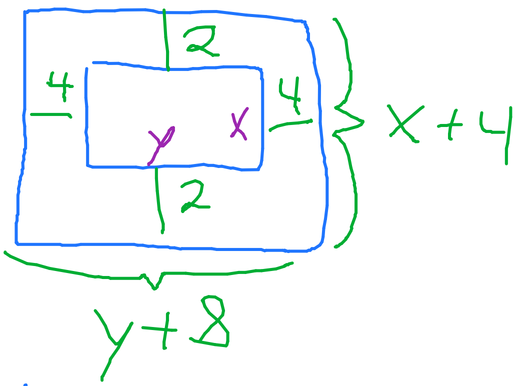
$\frac{10}{3} \times \frac{40}{3} \times \frac{40}{3}$

Volume $= \frac{1600}{27}$

HW 24.6: You are designing a paper with an area of 400 cm^2 to contain a printing area in the middle and have the margins of 2 cm at the top and bottom and 4 cm on each side. Find the largest possible printing area.

① Area

②



③ $A = xy$

④ $400 = (x+4)(y+8)$

⑤ If $x = 0$,

$$400 = (0+4)(y+8)$$
$$400 = 4(y+8)$$
$$100 = y+8$$
$$y = 92$$

If $y = 0$

$$400 = (x+4)(0+8)$$
$$400 = 8(x+4)$$
$$50 = x+4$$
$$x = 46$$

Domain $x: (0, 46)$

Domain $y: (0, 92)$

⑥ Solve ④ for y.

$$\frac{400}{x+4} = y+8$$
$$\frac{400}{x+4} - 8 = y$$

Plug y into ③

$$A = x\left(\frac{400}{x+4} - 8\right)$$
$$= \frac{400x}{x+4} - 8x$$
$$A' = \frac{1600}{(x+4)^2} - 8 = 0$$
$$\frac{1600}{(x+4)^2} = \frac{8}{1}$$
$$8(x+4)^2 = 1600$$
$$(x+4)^2 = 200$$
$$x+4 = \pm 10\sqrt{2}$$
$$x = -4 \cancel{\pm} 10\sqrt{2}$$

b/c ⑤

$$A' = 1600(x+4)^{-2} - 8$$
$$A'' = -3200(x+4)^{-3}$$

HW 24.6: You are designing a paper with an area of 400 cm^2 to contain a printing area in the middle and have the margins of 2 cm at the top and bottom and 4 cm on each side. Find the largest possible printing area.

$$A''(10\sqrt{2}-4) = \frac{-3200}{(10\sqrt{2})^3} < 0 \Rightarrow \text{Max}$$

7. $x = 10\sqrt{2} - 4$

$$y = \frac{400}{x+4} - 8 = \frac{400}{10\sqrt{2}-4+4} - 8 = 20\sqrt{2} - 8$$

$$A = xy = (10\sqrt{2}-4)(20\sqrt{2}-8) \simeq 206 \text{ cm}^3$$

Example 5: A Norman window is constructed by adjoining a semicircle to the top of an ordinary rectangular window. Find the dimensions of a Norman window of maximum area if the total perimeter is 10 feet.

① Area

②

③ $A = \frac{1}{2}\pi r^2 + 2rw$

④ $10 = P = \pi r + 2w + 2r$

⑤ If $r = 0$, $10 = 2w$

$w = 5$

If $w = 0$, $10 = \pi r + 2r$

$10 = r(\pi + 2)$

$r = \frac{10}{\pi + 2}$

<u>Domain r</u>: $\left(0, \frac{10}{\pi+2}\right)$

<u>Domain w</u>: $(0, 5)$

⑥ Solve ④ for w.

$10 - 2r - \pi r = 2w$

$\frac{10 - 2r - \pi r}{2} = w$

Plug w into ③

$A = \frac{1}{2}\pi r^2 + \cancel{2}r\left(\frac{10 - 2r - \pi r}{\cancel{2}}\right)$

$A = \frac{1}{2}\pi r^2 + 10r - 2r^2 - \pi r^2$

$A' = \frac{2\pi}{2} r + 10 - 4r - 2\pi r$

$= \pi r + 10 - 4r - 2\pi r$

$= 10 - 4r - \pi r$

$= 10 - r(4 + \pi) = 0$

$10 = r(4 + \pi)$

$r = \frac{10}{4 + \pi}$

$A'' = -(4 + \pi) < 0 \Rightarrow \max$

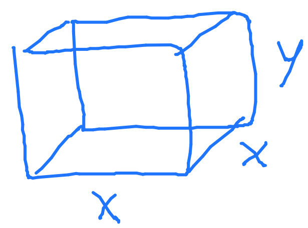
⑦ $r = \frac{10}{4 + \pi}$

$w = \frac{10}{4 + \pi}$

Example 6: A rectangular box is to have a square base and a volume of 800 ft^3. If the material for the base costs \$2 per square foot, the material for the sides costs \$4 per square foot, and the material for the top costs \$1 per square foot, determine the minimum cost for constructing such a box.

① Cost, C

②



③ $C = 2(x^2) + 4(4xy) + 1(x^2)$

$= 3x^2 + 16xy$

④ $800 = V = x^2 y$

⑤ Domain x & y:

$(0, \infty)$

⑥ Solve ④ for y.

$$\frac{800}{x^2} = y$$

Plug y in ③.

$$C = 3x^2 + 16x\left(\frac{800}{x^2}\right)$$

$$= 3x^2 + \frac{16(800)}{x}$$

$$= 3x^2 + 12800x^{-1}$$

$$C' = 6x - 12800x^{-2} = 0$$

$$6x - \frac{12800}{x^2} = 0$$

$$\frac{6x}{1} = \frac{12800}{x^2}$$

$$6x^3 = 12800$$

$$x^3 = 6400/3$$

$$x = 4 \cdot \sqrt[3]{100/3}$$

$$C'' = 6 + 25600x^{-3}$$

$$= 6 + \frac{25600}{x^3}$$

$$C''\left(4 \cdot \sqrt[3]{100/3}\right) > 0$$

$\Rightarrow$ min

⑦ $x = 4 \cdot \sqrt[3]{100/3}$

$$y = \frac{800}{16(100/3)^{2/3}}$$

$C = \$62.14$

Example 7: A company's marketing department has determined that if their product is sold at the price of p dollars per unit, they can sell $q = 2400 - 200p$ units. Each unit costs \$5 to make.

1. What price, p, should the company charge to maximize their revenue?

(1) Revenue

(2) N/A

(3) $R = \text{unitprice} \times \text{quantity}$
$= pq$

(4) $q = 2400 - 200p$

(5) Domain p: $[0, \infty)$

Cause you can sell them as freebies

(6) Plug (4) into (3)

$R = p(2400 - 200p)$
$= 2400p - 200p^2$

Take derivative set $= 0$.

$R' = 2400 - 400p = 0$

$p = 6$

$R'' = -400 < 0 \Rightarrow \max$

(7) $p = \$6$

Example 7: A company's marketing department has determined that if their product is sold at the price of p dollars per unit, they can sell $q = 2400 - 200p$ units. Each unit costs \$5 to make.

2. What price, p, should the company charge to maximize their profit?

① Profit

② N/A

③ $P = R - C$

$R = pq$

$C = 5q$

$= pq - 5q$

$= q(p-5)$

④ $q = 2400 - 200p$

⑤ Domain p: $[0, \infty)$

⑥ Plug ④ into ③.

$P = (2400 - 200p)(p-5)$

$= -200(p-12)(p-5)$

$= -200(p^2 - 17p + 60)$

Take derivative set $= 0$.

$P' = -200(2p - 17) = 0$

$p = 17/2 = \$8.50$

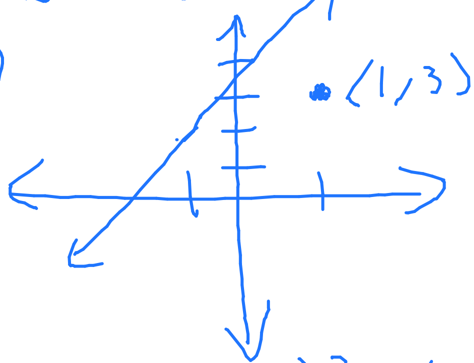
$P'' = -200(2) < 0 \Rightarrow \text{Max}$

⑦ $p = \$8.50$

Example 8: Find the point on the graph of $f(x) = 2x + 4$ that is the closest to the point $(1,3)$.

1 Distance $f(x)$

2



3 $D = (x-1)^2 + (y-3)^2$

(Maximizing $\sqrt{?}$ is equivalent to maximizing ?)

4 $y = 2x + 4$

5 Domain x,y: $(-\infty, \infty)$

(Look @ the graph)

6 Plug 4 into 3.

$$D = (x-1)^2 + (2x+4-3)^2$$
$$= (x-1)^2 + (2x+1)^2$$
$$= x^2 - 2x + 1 + 4x^2 + 4x + 1$$
$$= 5x^2 + 2x + 2$$

Take derivative set $= 0$.

$$D' = 10x + 2 = 0$$
$$x = -1/5$$

$D'' = 10 > 0 \Rightarrow \min$

7 $x = -1/5$

$y = 2x + 4 = \frac{18}{5}$

$\left(-\frac{1}{5}, \frac{18}{5}\right)$

HW 25.4: For a cylinder with a surface area of 20, what is the maximum volume that it can have? Recall that the volume of a cylinder is $\pi r^2 h$ and the surface area is $2\pi r h + 2\pi r^2$ where r is the radius and h is the height.

① Volume

②

③ $V = \pi r^2 h$

④ $20 = SA = 2\pi r h + 2\pi r^2$

$10 = \pi r h + \pi r^2$

⑤ If $r = 0$, $10 = 0$ ✗

Domain h: $(0, \infty)$

If $h = 0$, $10 = \pi r^2$

$r = \cancel{\pm}\sqrt{\frac{10}{\pi}}$

b/c real world problem

Domain r: $\left(0, \sqrt{\frac{10}{\pi}}\right)$

⑥ Solve ④ for h.

$10 - \pi r^2 = \pi r h$

$h = \frac{10 - \pi r^2}{\pi r}$

Plug h into 3.

$V = \cancel{\pi} r^{\cancel{2}}\left(\frac{10 - \pi r^2}{\cancel{\pi}\cancel{r}}\right)$

$V = r(10 - \pi r^2)$

$= 10r - \pi r^3$

Take derivative set $= 0$

$V' = 10 - 3\pi r^2 = 0$

$r = \sqrt{\frac{10}{3\pi}}$

$V'' = -6\pi r$

$V''\left(\sqrt{\frac{10}{3\pi}}\right) < 0 \Rightarrow \max$

⑦ $r = \sqrt{\frac{10}{3\pi}}$

$h = 2\sqrt{\frac{10}{3\pi}}$

$V = \pi r^2 h$

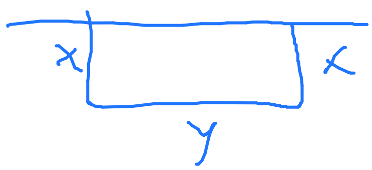
$= \cancel{\pi} \cdot \frac{10}{\cancel{3\pi}} \cdot 2\sqrt{\frac{10}{3\pi}}$

$= \frac{20}{3}\sqrt{\frac{10}{3\pi}}$

HW 26.1: A rectangular recreational field needs to be built outside of a gymnasium. Three walls of fencing are needed and the fourth wall is to be a wall of the gymnasium itself. The ideal area for such a field is exactly $490000 ft^2$. In order to minimize costs, it is necessary to construct the fencing using the least amount of material possible. Assuming that the material used in the fencing costs \$25/ft, what is the least amount of money needed to build this fence of ideal area?

1 Cost

2



3 $C = 25 \times \text{perimeter}$

$\downarrow \; P = 2x + y$

$= 25(2x+y)$

4 $490000 = A = xy$

5 Domain x, y: $(0, \infty)$

6 Solve 4 for y.

$$y = \frac{490000}{x}$$

Plug y into 3

$$C = 25(2x + 490000x^{-1})$$

Take derivative set $= 0$.

$$C' = 25(2 - 490000x^{-2}) = 0$$

$$2 - \frac{490000}{x^2} = 0$$

$$\frac{2}{1} = \frac{490000}{x^2}$$

$$2x^2 = 490000$$

$$x = 350\sqrt{2}$$

$$C'' = 25(-(-2)490000x^{-3})$$

$$= \frac{50(490000)}{x^3}$$

$C''(350) > 0 \Rightarrow \text{min}$

7 $x = 350\sqrt{2}$

$$y = \frac{490000}{350\sqrt{2}}$$

$$= 700\sqrt{2}$$

$$C = 25(2 \cdot 350\sqrt{2} + 700\sqrt{2})$$

$$\approx \$98994.95$$