Lesson 28: More on Antiderivatives and Indefinite Integration

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A differential eqn in x and y is an eqn that relates x, y, and y'.

examples: $y' = 3x$

$x + y' = 5$

$2y + xy = e^x$

Ex 3: Solve the differential eqn $y' = 3x$

$$\int y'\,dx = \int 3x\,dx$$

$$\int \frac{dy}{\cancel{dx}}\cancel{dx} = \int 3x\,dx$$

$$\int dy = \int 3x\,dx$$

$$y = \frac{3x^2}{2} + C$$

$$\boxed{\text{Recall} \quad y' = \frac{dy}{dx}}$$

The solution for Ex 3

$$y = \frac{3}{2}x^2 + C$$

is called the general solution.

But what if I give you a y-value at some x?

We call that the initial condition.

The answer of a specific function is particular solution

A differential eqn w/ initial condition is an initial value problem (IVP).

Ex 4: Solve IVP $y' = 3x$ with $y(0) = 2$

By Ex 3, $y = \frac{3}{2}x^2 + C$

So $2 = y(0) = \frac{3}{2}(0)^2 + C$

$2 = 0 + C$

$2 = C$

Answer: $y = \frac{3}{2}x^2 + 2$

Ex 5: Solve the IVP

$\boxed{2y'' = e^x + 4}$ $\quad y'(0) = 5 \quad y(2) = 10$

Solve for y''

$$y'' = \frac{e^x + 4}{2} = \frac{e^x}{2} + \frac{4}{2} = \frac{e^x}{2} + 2$$

Integrate each side

$$\int y'' dx = \int \left(\frac{e^x}{2} + 2\right) dx$$

$$y' = \frac{e^x}{2} + 2x + c$$

Ex 5: Solve the IVP

$$2y'' = e^x + 4 \quad \boxed{y'(0) = 5} \quad y(2) = 10$$

Plug the condition into y'

$$y' = \frac{e^x}{2} + 2x + c$$

$$5 = y'(0) = \frac{e^0}{2} + 2(0) + c$$

$$5 = \frac{1}{2} + 0 = c$$

$$c = 5 - \frac{1}{2} = \frac{9}{2}$$

So $y' = \frac{e^x}{2} + 2x + \frac{9}{2}$

Repeat again

$$\int y' dx = \int \left(\frac{e^x}{2} + 2x + \frac{9}{2}\right) dx$$

$$y = \frac{e^x}{2} + \frac{2x^2}{2} + \frac{9}{2}x + c$$

$$y = \frac{e^x}{2} + x^2 + \frac{9}{2}x + c$$

Ex 5: Solve the IVP

$$2y'' = e^x + 4 \qquad y'(0) = 5 \qquad \boxed{y(2) = 10}$$

Plug the condition into y.

$$y = \frac{e^x}{2} + x^2 + \frac{9}{2}x + c$$

$$10 = y(2) = \frac{e^2}{2} + (2)^2 + \frac{9}{\cancel{2}}\cancel{(2)} + c$$

$$10 = \frac{e^2}{2} + 4 + 9 + c$$

$$10 = \frac{e^2}{2} + 13 + c$$

$$c = -3 - e^2/2$$

Final Answer

$$y = \frac{e^x}{2} + x^2 + \frac{9}{2}x - 3 - \frac{e^2}{2}$$

Ex 6: The rate of growth $\frac{dP}{dt}$ of a population of bacteria is proportional to the square root of t with a constant coefficient of 2, where P is the population size and t is the time in days $(0 \le t \le 10)$ The initial size of the population is 500. Approximate the population after 7 days. Round the answer to the nearest integer.

$$\frac{dP}{dt} = 2\sqrt{t} = 2t^{1/2}$$

$$\int \frac{dP}{dt}\,dt = \int 2t^{1/2}\,dt$$

$$P = 2 \cdot \frac{2}{3} t^{3/2} + C$$

$$= \frac{4}{3} t^{3/2} + C$$

$$P(0) = 500$$

$$500 = P(0) = \frac{4}{3}(0)^{4/3} + C$$

$$500 = C$$

$$P = \frac{4}{3} t^{3/2} + 500$$

Note that I'm not asked for P, but for P(7).

$$P = \frac{4}{3} t^{3/2} + 500$$

$$P(7) = \frac{4}{3}(7)^{3/2} + 500 \approx 525$$

Ex 7: A hot air balloon is rising vertically with a velocity of 16 feet per second. A very small ball is released from the hot air balloon at the instant when it is 64 feet above the ground. Use $a(t) = -32\ ft/sec^2$ as the acceleration due to gravity.

So $v(0) = 16$ and $s(0) = 64$

a) How many seconds after its release will the ball strike the ground? $\rightarrow \boxed{s(t) = 0}$

Recall $s'(t) = v(t)$ and $v'(t) = a(t)$.

$$v(t) = \int a(t)\,dt = \int -32\,dt = -32t + C$$

$$16 = v(0) = -32(0) + C$$

$$16 = C$$

$$v(t) = -32t + 16$$

$$s(t) = \int v(t)\,dt = \int (-32t + 16)\,dt$$
$$= -\frac{32t^2}{2} + 16t + C$$
$$= -16t^2 + 16t + C$$
$$64 = s(0) = -16(0)^2 + 16(0) + C$$
$$64 = C$$

Remember I want to solve s(t) = 0 for t.

$$s(t) = -16t^2 + 16t + 64 = 0$$
$$-16(t^2 - t - 4) = 0$$
$$t = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-4)}}{2} = \frac{1 \pm \sqrt{17}}{2}$$

$t = \frac{1+\sqrt{17}}{2} \approx 2.56$ sec $\quad$ $\cancel{t = \frac{1-\sqrt{17}}{2}}$ b/c neg.

b) At what velocity will it hit the ground?

Plug the answer from (a) into v(t).

$$v(t) = -32t + 16$$
$$v(2.56) = -32(2.56) + 16$$
$$= -65.92 \text{ ft/sec}$$