Lesson 34: Numerical Integration

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With the FTC, we are able to evaluate definite integrals for certain integrands.

However, there are many functions that we do know how to integrate

ex. ① $f(x) = e^x\sqrt{x^2+1}$

② $f(x) = \dfrac{\sin x}{x+1}$

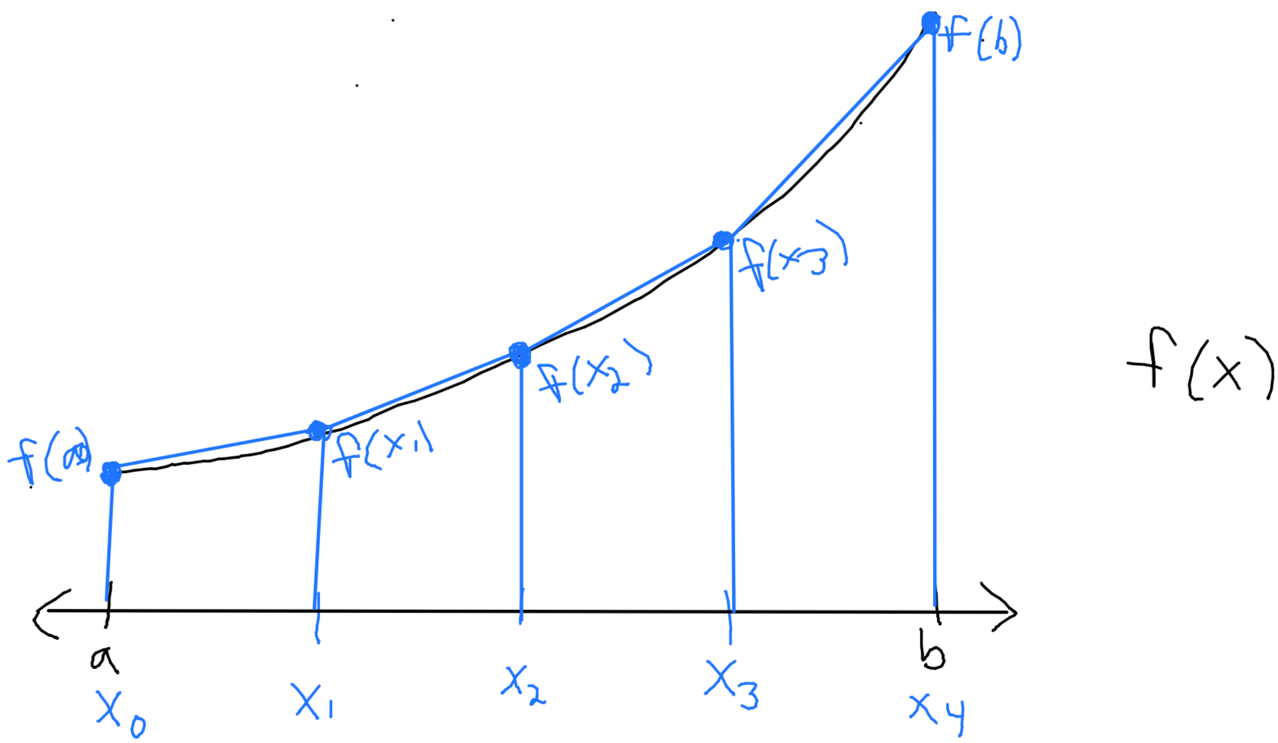
The Trapezoid Rule is an approx method that allows us to approx definite integrals.

Trapezoid Rule is similar to Riemann Sums

Instead of using rectangles, we are using trapezoids. (Shocking right)

Suppose $f(x)$ is continuous on $[a,b]$. We want to approx the area

$$\int_a^b f(x)\,dx$$ using 4 trapezoids



Recall the area of a trapezoid is

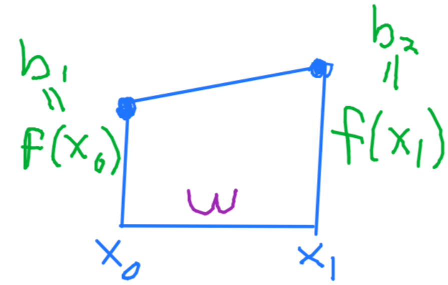
$$A = \frac{1}{2}(b_1 + b_2)w$$

Good news: The width of each sub-intervals is the same as in Riemann sums

$$\Delta x = \frac{b-a}{n} = w$$

Let's determine the bases of the first trapezoid

$b_1 = f(x_0)$ and $b_2 = f(x_1)$



Area of 1st trapezoid, t_1, is

$$t_1 = \frac{1}{2}\left(f(x_0) + f(x_1)\right)\Delta x$$

Similarly,

$$t_2 = \frac{1}{2}\left(f(x_1) + f(x_2)\right)\Delta x$$

$$t_3 = \frac{1}{2}\left(f(x_2) + f(x_3)\right)\Delta x$$

$$t_4 = \frac{1}{2}\left(f(x_3) + f(x_4)\right)\Delta x$$

Let's sum at four of these.

$$\begin{aligned} t_1 + t_2 + t_3 + t_4 &= \frac{1}{2}\left(f(x_0) + f(x_1)\right)\Delta x \\ &\quad + \frac{1}{2}\left(f(x_1) + f(x_2)\right)\Delta x \\ &\quad + \frac{1}{2}\left(f(x_2) + f(x_3)\right)\Delta x \\ &\quad + \frac{1}{2}\left(f(x_3) + f(x_4)\right)\Delta x \\ &= \frac{1}{2}\begin{pmatrix} f(x_0) + f(x_1) + f(x_1) + f(x_2) \\ + f(x_2) + f(x_3) + f(x_3) + f(x_4) \end{pmatrix}\Delta x \\ &= \frac{1}{2}\left(f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)\right)\Delta x \end{aligned}$$

We can extend this analysis to n trapezoids.

$$T_n = \frac{1}{2}\Delta x\left(f(x_0) + 2f(x_1) + \ldots + 2f(x_{n-1}) + f(x_n)\right)$$

Where $x_i = a + i\Delta x$