Lesson 35: Exponential Growth

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Ex 1: Solve IVP $\frac{dy}{dt} = 2y$ with $y(0) = 100$

Try getting all y terms to one side.

$$\frac{dy}{y} = 2dt$$

$$\int \frac{dy}{y} = \int 2dt$$

$$e^{\ln y} = e^{2t+C}$$

$$y = e^{2t+C}$$

$$= e^{2t} \cdot \underbrace{e^{C}}_{\text{constant}}$$

$$= Ce^{2t}$$

Ex 1: Solve IVP $\frac{dy}{dt} = 2y$ with $y(0) = 100$

Now with $y = Ce^{2t}$ use

$$100 = Ce^{2(0)} = C$$

So $\boxed{y = 100e^{2t}}$

Exponential Growth Model

If y is a differential function of t such that

$$\frac{dy}{dt} = y' = ky \text{ for some constant } k$$

then $y = Ce^{kt}$ where C is a constant.

k - proportionality constant or growth rate

C - the initial value of y

If $k > 0$ and $C > 0$, then this model is called the exponential growth model.

Ex 2: The rate of change of a population P is proportional to P (use k for the proportionality constant). Answer the following questions.

(a) What is dP/dt?

$$\frac{dP}{dt} = kP$$

(b) Find P.

$$P = Ce^{kt}$$

(c) If P = 200 when t = 0 and P = 400 when t = 2, what is P(4)?

$$200 = Ce^{k(0)} = C$$

$$P = 200e^{kt}$$

$$400 = 200e^{2k}$$

$$2 = e^{2k}$$

$$\ln(2) = 2k$$

$$k = \frac{\ln(2)}{2}$$

$$P = 200\exp\left[\frac{\ln(2)}{2}t\right]$$

$$P(4) = 200\exp\left[\frac{\ln(2)}{2} \cdot 4\right]$$

$$= 200 \exp[2 \ln 2]$$
$$= 200 \cancel{\exp}[\cancel{\ln} 2^2]$$
$$= 200 \cdot 4 = 800$$

(d) If P = 200 when t = 1 and P = 400 when t = 2, what is P(4)?

$$200 = Ce^k \quad (1)$$
$$400 = Ce^{2k} = \underbrace{Ce^k}e^k \quad (2)$$

Plug (1) into

$$400 = 200e^k$$
$$2 = e^k$$
$$\ln 2 = k$$

Plug k into (1).

$$200 = C\cancel{e}^{\cancel{\ln} 2} = 2C$$
$$C = 100$$
$$P = 100 \exp[\ln(2) t]$$
$$P(4) = 100 \exp[\ln(2) \cdot 4]$$
$$= 100 \cancel{\exp}[\cancel{\ln}(2^4)]$$
$$= 100(16) = 1600$$

Ex 3: In a savings account where the interest is compounded continuously, if the initial investment is 500 dollars and the annual interest rate is 3%, how much money will there be in 10 years?

Note $C = 500 > 0$ and $k = 0.03 > 0$. So

$$y = 500 \exp[0.03t]$$

$$y(10) = 500\exp[0.03(10)]$$
$$= 500\exp[0.3]$$
$$\approx \boxed{\$674.93}$$

Ex 3 (Continuation) How long does it take to double the initial investment?

Previously we found $y = 500\exp[0.03t]$ . .

Double the initial investment $\Rightarrow 2(500)$. So,

$$2(\cancel{500}) = \cancel{500}\exp[0.03t]$$
$$\ln 2 = 0.03t$$
$$t = \frac{\ln 2}{0.03} \approx \boxed{23.1 \text{ years}}$$

Ex 4: In a savings account where the interest is compounded continuously, if the initial investment is 100 dollars and there are 150 dollars in 8 years, what is the annual interest rate?

Solving for k.

$$\left.\begin{matrix} k - \text{interest rate} > 0 \\ C = 100 > 0 \end{matrix}\right\} y = 100e^{kt}$$

The question also states $y(8) = 150$

$$150 = 100e^{k(8)}$$
$$\frac{3}{2} = \frac{150}{100} = e^{8k}$$

$$\ln(3/2) = 8k$$
$$k = \frac{1}{8}\ln\left(\frac{3}{2}\right) \approx 0.05$$
$$\Rightarrow \boxed{5\%}$$