Lesson 36: Exponential Decay

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Recall Exponential Growth

$$y = Ce^{kt}$$

where $C > 0$ and $k > 0$

Exponential Decay is very similar to Exponential Growth.

What's different?

Exponential Decay has $k < 0$

Exponential Decay Model

If y is a differential function of t such that

$$\frac{dy}{dt} = y' = ky \quad \text{for some constant } k$$

then $y = Ce^{kt}$ where C is a constant.

k - proportionality constant or growth rate

C - the initial value of y

If $k < 0$ and $C > 0$, then this model is called the exponential decay model.

Radioactive Isotopes and Half Life:

- Radioactive isotopes decay over time

- It follows the exponential decay model

- Decay rate is distinct for each isotope
  - Characterized by half-life

i.e. Half-life of a radioactive isotope is the time that it takes for the isotope to reduce to half of its original quantity.

Example: If ^{226}Ra has half-life of 1599, it means it takes 1599 to reduce to half its original quantity.

Radioactive Isotopes and Half Life:

Since the decay rate of a radioactive isotope is characterized by half-life, the constant k in the decay model must have some connection to half-life. What is the connection then?

$$y = C\exp[kt]$$

By the definition of half-life,

$$\frac{1}{2}\cancel{C} = \cancel{C}\exp[k(\text{half-life})]$$

$$\ln(1/2) = k(\text{half-life})$$

$$k = \frac{\ln(1/2)}{\text{half-life}} = \frac{-\ln(2)}{\text{half-life}} < 0 \text{ always}$$

Ex 1: The population of a country follows exponential growth and the continuous annual rate of change k of the population is -0.001. The population is 10 million when t = 2. What is the population when t = 6?

$$k = -0.001$$

$$y = C\exp[-0.001t]$$

$$y(2) = 10$$
$$10 = C\exp[-0.001(2)]$$
$$= C\exp[-0.002]$$
$$10\exp[0.002] = C$$

$$y = 10e^{0.002}e^{-0.001t}$$
$$y = 10\exp[0.002 - 0.001t]$$
$$y(6) = 10\exp[0.002 - 0.001(6)]$$
$$= 10\exp[-0.004]$$
$$\approx \boxed{9.9601} \text{ millions}$$

Ex 2: The radioactive isotope ^{226}Ra has a half-life of 1599 years. If there are 10 grams of ^{226}Ra initially, how much is there after 1,000 years?

$$y = C\exp[kt]$$

Recall $k = -\frac{\ln(2)}{\text{half-life}} = -\frac{\ln(2)}{1599}$ & $C = 10$

So $y = 10\exp\left[-\frac{\ln(2)}{1599}t\right]$

$$y(1000) = 10\exp\left[-\frac{\ln(2)}{1599}(1000)\right] \approx \boxed{6.4828 \text{ grams}}$$

Ex 3: The radioactive isotope ^{14}C has a half-life of 5715 years. If there are 1.6 grams left after 1,000 years, how much is the initial quantity?

$$y = C\exp[kt]$$

Recall $k = -\frac{\ln(2)}{\text{half-life}} = -\frac{\ln(2)}{5715}$

So $y = C\exp\left[-\frac{\ln(2)}{5715}t\right]$.

Note $y(1000) = 1.6$

$$1.6 = C\exp\left[-\frac{\ln(2)}{5715}(1000)\right]$$

$$C = 1.6\exp\left[\frac{\ln(2)}{5715}(1000)\right]$$

$$\approx \boxed{1.8063 \text{ g}}$$

Ex 3 (Continuation) How much is there after 10,000 years?

From previous part,

$$C = 1.6\exp\left[\frac{\ln(2)}{5715}(1000)\right]$$

$$y = C\exp\left[-\frac{\ln 2}{5715}t\right]$$

Putting them together

$$y = 1.6\exp\left[\frac{\ln 2}{5715}(1000)\right]\cdot\exp\left[-\frac{\ln 2}{5715}t\right]$$

$$= 1.6\exp\left[\frac{\ln 2}{5715}(1000) - \frac{\ln 2}{5715}t\right]$$

$$= 1.6\exp\left[\frac{\ln 2}{5715}(1000 - t)\right]$$

$$y(10{,}000) = 1.6\exp\left[\frac{\ln 2}{5715}(1000 - 10{,}000)\right]$$

$$\approx \boxed{0.5371}$$

HW 36.7: Radioactive radium has a half-life of approx 1599 years. What percent of a given amount remains after 300 years?

$$k = \frac{\ln(1/2)}{1599} = \frac{-\ln(2)}{1599} \Rightarrow y = C\exp\left[\frac{-\ln(2)}{1599}t\right]$$

$$y(300) = C\exp\left[-\frac{\ln(2)}{1599}(300)\right]$$

$$= \underbrace{0.8781}C$$

$\Downarrow$

87.81% remains