Reminders

- NEXT CLASS QUIZ 5 on
  - Volume of Revolutions
    - Disks (Lesson 14)
    - Washers (Lesson 15)
- FRIDAY – NEXT MONDAY => NO CLASS

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MA 16020: Lesson 17
Volume By Revolution
Shell Method

By Alexandra Cuadra

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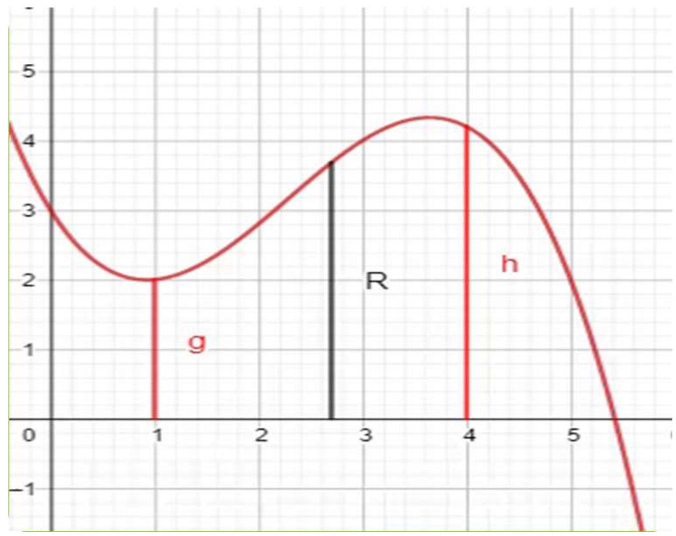
So far…

- We have learned how to find the volume of a solid of revolution by integrating
  - In the same way, we calculate the area under a curve
    - Running a line segment of varying length across the region, and adding them up



https://www.geogebra.org/m/tgceabb2#material/tnnhu7gz

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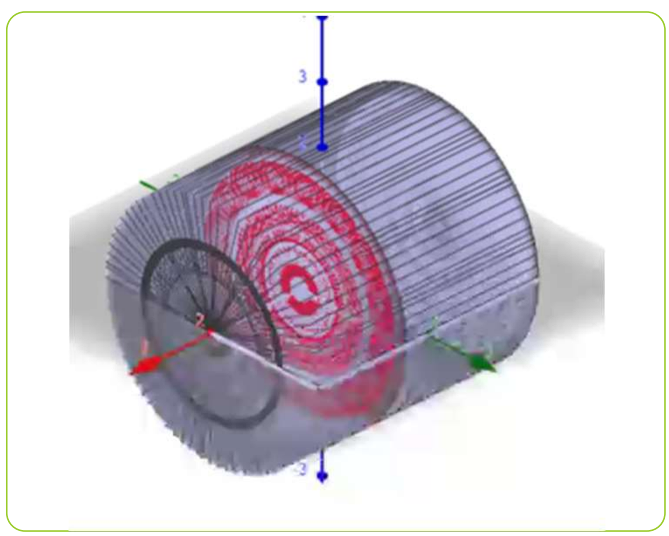
In other words,

- We learned to find the volume of a solid of revolution by
  - Running some area across a shape and add them up.
  - Like in the case of the cylinder shown on the right.



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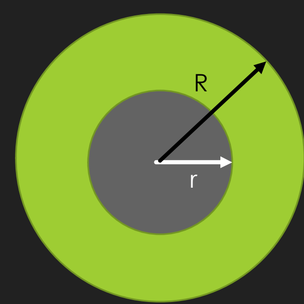
But what were those "shapes"?
ANSWER: CROSS-SECTIONS

Sometimes it was a disk

Whose area is πR^2

Sometimes it was a washer



Whose area is $\pi(R^2 - r^2)$

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In Today's Lecture, we will be covering the case, when neither method (Disk nor Washer) is easy.

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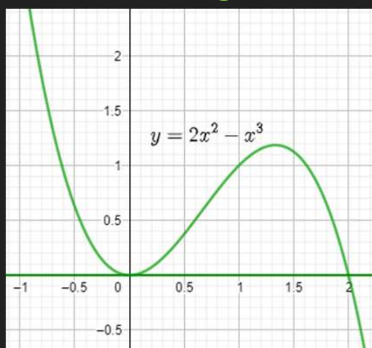
Example 1: Find the volume obtained by revolving the region bounded by the curves

$$y = 2x^2 - x^3 \quad \text{and} \quad y = 0$$

About the y-axis.

Draw the region.



https://www.geogebra.org/m/jqfyndpu

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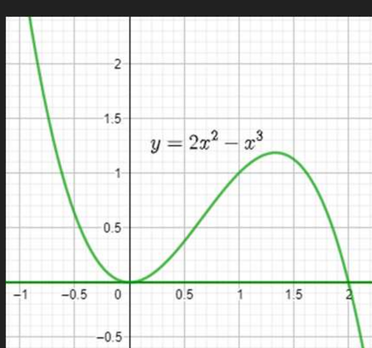
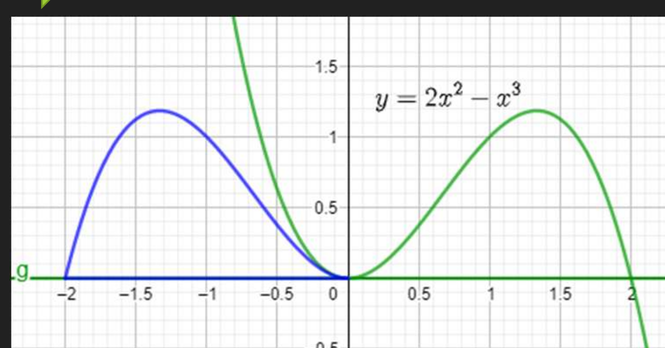
Example 1: Find the volume obtained by revolving the region bounded by the curves

$$y = 2x^2 - x^3 \quad \text{and} \quad y = 0$$

About the y-axis.

Rotation about y-axis





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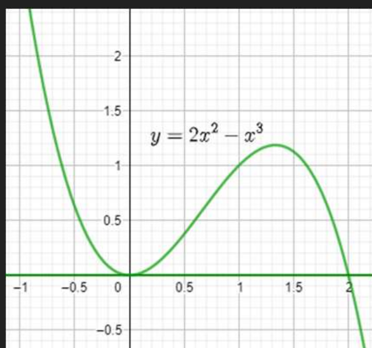
Example 1: Find the volume obtained by revolving the region bounded by the curves

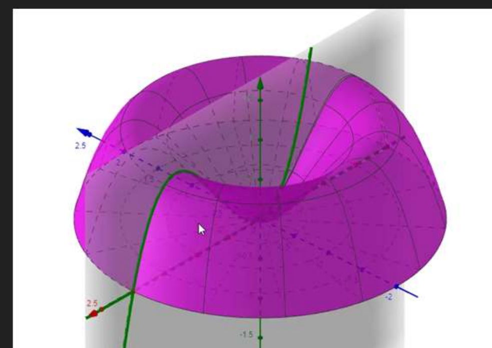
$$y = 2x^2 - x^3 \quad \text{and} \quad y = 0$$

About the y-axis.



WASHER ???

Furthermore, 3-D



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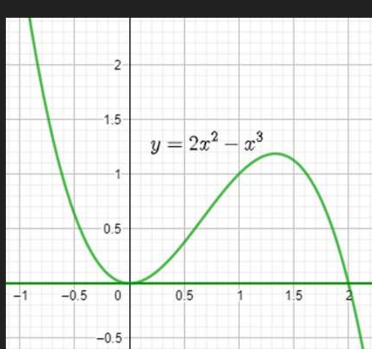
Example 1: Find the volume obtained by revolving the region bounded by the curves

$$y = 2x^2 - x^3 \quad \text{and} \quad y = 0$$

About the y-axis.



Technically, yes. It is a Washer Problem.
But there are two issues:

1. Given we are revolving around y-axis, we want to solve our equations for x.

   i.e. Solve $y = 2x^2 - x^3$ for x.

   But that is easier said than done.

2. For washer problems, we need two equations for each radius.

   Here we have both radius depend on the same function.

https://www.geogebra.org/m/jqfyndpu

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So how can I do this kind of problem without giving myself a headache?

ANSWER: SHELL METHOD

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GEOMETRY: Finding The Volume of A Hollow Cylinder

- To find the volume of this hollow cylinder, we used the same idea when washers were first introduced.

$$V_{total} = V_{outer} - V_{inner}$$

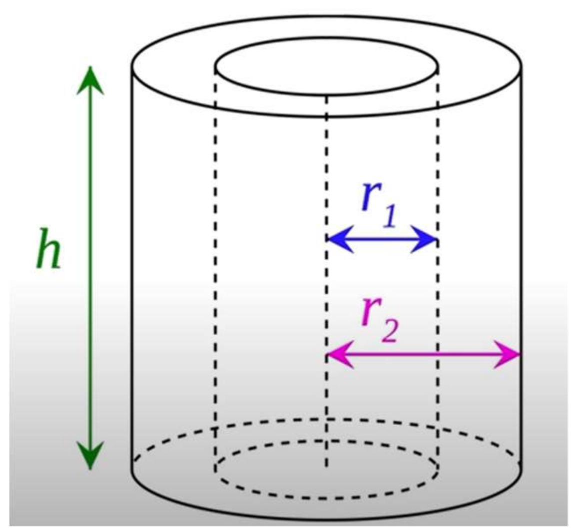
- Remember the volume of a cylinder is $\pi r^2 h$. So

$V_{outer} = \pi(r_2)^2 h$ and $V_{inner} = \pi(r_1)^2 h$

- Hence $V_{total} = \pi r_2^2 h - \pi r_1^2 h$

$$= \pi h(r_2^2 - r_1^2)$$

$$= \pi h(r_2 - r_1)(r_2 + r_1)$$



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GEOMETRY: Finding The Volume of A Hollow Cylinder

So let's be clever

- Let's take the sum $r_2 + r_1$ and express it as an average.

  i.e. $(r_1 + r_2)/2$

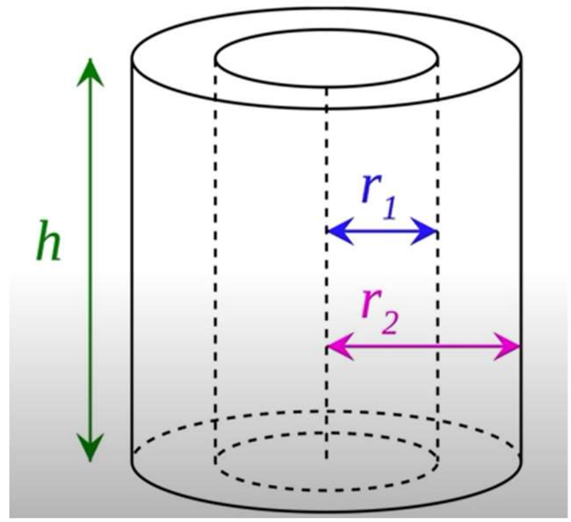
- To do that multiple the equation below by 2/2.

  $$V_{total} = \pi h(r_2 - r_1)(r_2 + r_1)$$

  $$= 2\pi h(r_2 - r_1)\left(\frac{r_2+r_1}{2}\right)$$

- Since we have the average radius in our equation, we can now call $r = \frac{r_1+r_2}{2}$.

  $$V_{total} = 2\pi h(r_2 - r_1)r$$



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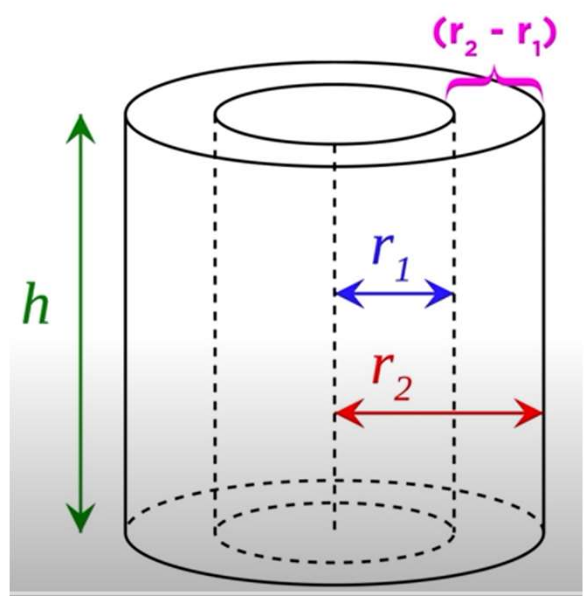
GEOMETRY: Finding The Volume of A Hollow Cylinder

- Note that the difference of the radii gives us the thickness of the cylinder.
  - Let Δr be that difference

    $$\Delta r = r_2 - r_1$$

- Hence we can say that the volume of the hollow cylinder is

  $$V_{total} = 2\pi rh \cdot \Delta r$$



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GEOMETRY: Finding The Volume of A Hollow Cylinder

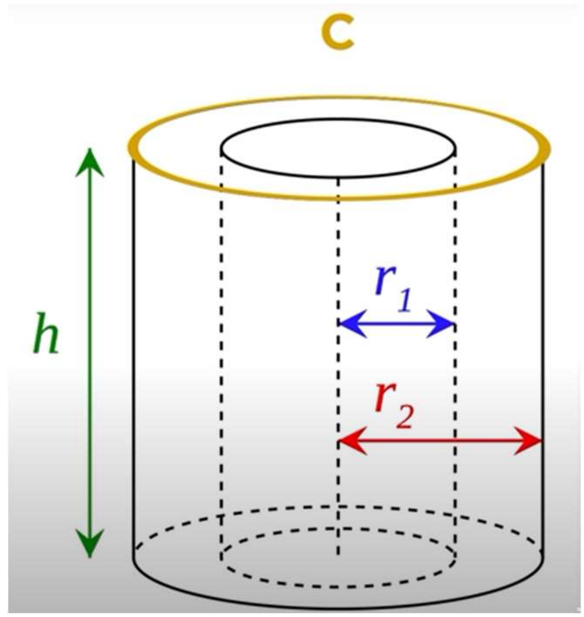
- One way to remember this

$$V_{total} = 2\pi rh \cdot \Delta r$$

is to see that $2\pi r$ is the same as the circumference, C, (as shown in the image) of the cylinder.

- So this is just the

circumference × height × thickness.



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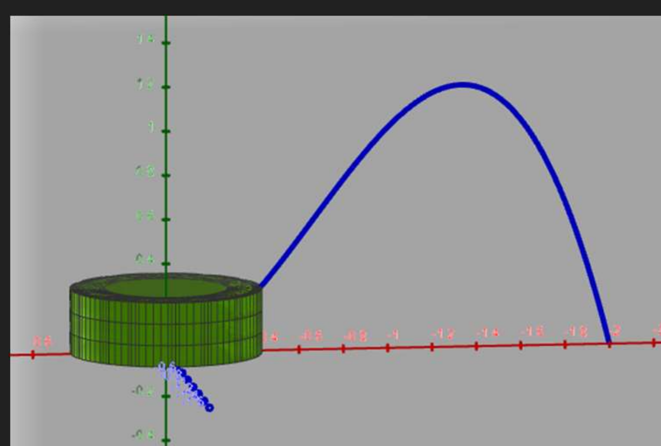
So how does this help us answer Example 1?

- The reason is USEFUL is that
  - For this problem, we no longer have to solve for x in terms of y.
- If we picture one possible shell, it will have a
  - $height = f(x)$
  - $circumference = 2\pi x$
- As this shell spans the volume, we then have

$$V = \int_a^b 2\pi x \cdot f(x) dx$$



https://www.geogebra.org/m/jqfyndpu

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Example 1: Find the volume obtained by revolving the region bounded by the curves

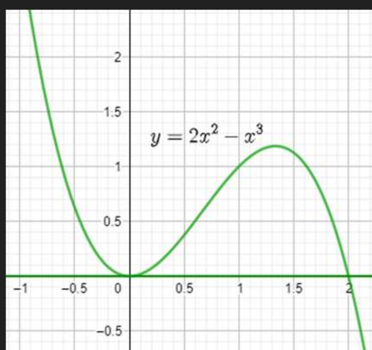
$$y = 2x^2 - x^3 \quad \text{and} \quad y = 0$$

About the y-axis. $\Rightarrow dx$ problem

$$V = 2\pi \int_0^2 x(2x^2 - x^3)\,dx$$

$$= 2\pi \int_0^2 (2x^3 - x^4)\,dx$$

$$= 2\pi \left(\frac{2x^4}{4} - \frac{x^5}{5}\right)\Big]_0^2 = \frac{16\pi}{5}$$



https://www.geogebra.org/m/jqfyndpu

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Shell Method Formula

Since we are just cutting out parallel to the axis, we choose dx or dy in the following way:

- Rotating around y-axis
  $\Rightarrow$ " dx " problem

$$V = 2\pi \int_a^b x \cdot f(x)\,dx$$

- Rotating around x-axis
  $\Rightarrow$ " dy " problem

$$V = 2\pi \int_c^d y \cdot g(y)\,dy$$

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Example 2: Find the volume obtained by revolving the region bounded by the curves

$$y = x^3 - 7x^2 + 15x - 9, \quad y = 0, \quad \text{and} \quad 1 \le x \le 3$$

About the y-axis.

$\Rightarrow dx$ problem if done w/ shells

$$V = 2\pi \int_1^3 x(x^3 - 7x^2 + 15x - 9)\,dx$$

$$= 2\pi \int_1^3 (x^4 - 7x^3 + 15x^2 - 9x)\,dx$$

$$= 2\pi \left(\frac{x^5}{5} - \frac{7x^4}{4} - \frac{15x^3}{3} - \frac{9x^2}{2}\right)\Big]_1^3 = \frac{332\pi}{15}$$

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Example 3: Find the volume obtained by revolving the region bounded by the curves

$$y = \sin(x^2) \quad \text{where} \quad 0 \le x \le 1$$

About the y-axis.

$\Rightarrow dx$ problem if done w/ shells

$$V = 2\pi \int_0^1 x\sin(x^2)\,dx$$

$$\frac{u = x^2}{du = 2x\,dx} \qquad 2\pi \int x\sin(u)\,\frac{du}{2x}$$

$$\frac{du}{2x} = dx$$

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Example 3: Find the volume obtained by revolving the region bounded by the curves

$$y = \sin(x^2) \quad \text{where} \quad 0 \le x \le 1$$

About the y-axis.

$V = \pi \int \sin(u)\, du$

$= \pi(-\cos(u))$

$= -\pi \cos(x^2)\Big]_0^1$

$= -\pi[\cos(1) - \cos(0)]$

$= -\pi[\cos(1) - 1] \approx 1.4442$

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Example 4: Find the volume obtained by revolving the region bounded by the curves

$$x = y - y^2 \quad \text{and} \quad x = 0$$

About the x-axis.

$\Rightarrow$ dy problem if done w/ shells

Let's find the bounds

$0 = y - y^2$

$0 = y(1-y)$

$y = 0, 1$

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Example 4: Find the volume obtained by revolving the region bounded by the curves

$$x = y - y^2 \qquad \text{and} \qquad x = 0$$

About the x-axis.

$$V = 2\pi \int_0^1 y(y - y^2)\,dy$$

$$= 2\pi \int_0^1 (y^2 - y^3)\,dy$$

$$= 2\pi \left(\frac{y^3}{3} - \frac{y^4}{4}\right)\Big]_0^1 = \frac{\pi}{6}$$

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Example 5: Find the volume obtained by revolving the region bounded by the curves

$$x = e^{y^2} \qquad \text{and} \qquad 0 \le y \le 2$$

About the x-axis.

⇒ dy problem if done w/ shells

$$V = 2\pi \int_0^2 y e^{y^2}\,dy$$

$u = y^2$

$du = 2y\,dy$

$\frac{du}{2y} = dy$

$$2\pi \int y e^u \frac{du}{2y}$$

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Example 5: Find the volume obtained by revolving the region bounded by the curves

$$x = e^{y^2} \qquad \text{and} \qquad 0 \leq y \leq 2$$

About the x-axis.

$$V = \pi \int e^u \, du$$

$$= \pi e^u$$

$$= \pi e^{y^2} \Big]_0^2$$

$$= \pi (e^{2^2} - e^{0^2}) = \pi (e^4 - 1)$$

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When do we apply Disk Method or Washer Method or Shell Method?

- When the region "hugs" the axis of rotation
  ⇒ Disk Method
- When there is a "gap" between the region and axis of rotation
  ⇒ Washer Method
- But if you find solving for x or y, in either method, is hard
  ⇒ Shell Method

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Formulas from Lessons 14 and 15 and 17

For rotation around x-axis:

- Disk Method:

$$V = \pi \int_a^b [f(x)]^2 \, dx$$

- Washer Method:

$$V = \pi \int_a^b (R^2 - r^2) \, dx$$

- Shell Method:

$$V = 2\pi \int_c^d y \cdot g(y) \, dy$$

For rotation around y-axis:

- Disk Method:

$$V = \pi \int_c^d [g(y)]^2 \, dy$$

- Washer Method:

$$V = \pi \int_c^d (R^2 - r^2) \, dy$$

- Shell Method:

$$V = 2\pi \int_a^b x \cdot f(x) \, dx$$

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Formulas from Lesson 16 Rotation around any non-Axis Formulas

For rotation around the line $y = e$:

- Disk Method:

$$V = \pi \int_a^b [f(x) - e]^2 \, dx$$

- Washer Method:

$$V = \pi \int_a^b \left((R - e)^2 - (r - e)^2\right) dx$$

For rotation around the line $x = e$:

- Disk Method:

$$V = \pi \int_c^d [g(y) - e]^2 \, dy$$

- Washer Method:

$$V = \pi \int_c^d \left((R - e)^2 - (r - e)^2\right) dy$$

Note: That these formulas work for the case of x-axis ($y = 0$) and y-axis ($x = 0$).

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GeoGebra Link for Lesson 17

- https://www.geogebra.org/m/f3wrypfh
- Note click on the play buttons on the left-most screen and the animation will play/pause.

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