

Lesson 21: First-Order Linear Differential Equations I

Warm-Up: Compute the following derivatives:

(a) $\frac{d}{dt} \left(\int P(t) dt \right)$

By the Fundamental Theorem of Calculus (FTC),

$$\frac{d}{dt} \left(\int P(t) dt \right) = P(t)$$

(b) $\frac{d}{dt} \left(e^{\int SP(t) dt} \right)$

By the Chain Rule,

$$\frac{d}{dt} \left(e^{\int SP(t) dt} \right) = e^{\int SP(t) dt} \cdot \frac{d}{dt} \left(\int SP(t) dt \right)$$

by (a)

$$e^{\int SP(t) dt} \cdot P(t)$$

(c) $\frac{d}{dt} \left(y(t) e^{\int SP(t) dt} \right)$

By the Product Rule,

$$\frac{d}{dt} \left(y(t) e^{\int SP(t) dt} \right) = y'(t) e^{\int SP(t) dt} + y(t) \cdot \frac{d}{dt} \left(e^{\int SP(t) dt} \right)$$

by (b)

$$y'(t) e^{\int SP(t) dt} + y(t) P(t) e^{\int SP(t) dt}$$

Let's define what a First-Order Linear Differential Equation

- First-order means that only the first derivative appears (so, no y'' , y''' , etc)

- Linear means that y' and y are not multiplied together in any combination.

example: $y' + ty = t^2 + 6$

Not example: $yy' + y = 1$

- Lesson 8.1: First-Order Linear
- Differential Equation is an equation that relates one or more functions and their derivatives.

A first order linear equation can be written in the standard form:

$$y' + P(t)y = Q(t) \quad (*)$$

Why do we want it in this form? To do the following

Let $u(t) = \exp[\int P(t) dt]$. Multiply both sides of (*) by $u(t)$, we get

$$y' \exp[\int P(t) dt] + P(t)y \exp[\int P(t) dt] = Q(t) \exp[\int P(t) dt]$$

By Warm-Up ③, the LHS is

$$\frac{d}{dt} (y \exp[\int P(t) dt]) = Q(t) \exp[\int P(t) dt]$$

Remember $u(t) = \exp[\int P(t) dt]$

$$\frac{d}{dt} (y \cdot u(t)) = Q(t) u(t)$$

Integrate both sides by dt

$$\int \frac{d}{dt} (y \cdot u(t)) dt = \int Q(t) u(t) dt$$

By the FTC,

$$y \cdot u(t) = \int Q(t) u(t) dt$$

Definition: The term $u(t) = \exp[\int P(t) dt]$ is called an integrating factor.

To summarize:

Given an equation of the form

$$y' + P(t)y = Q(t)$$

a solution is given by

$$y \cdot u(t) = \int Q(t) u(t) dt$$

where $u(t) = \exp[\int P(t) dt]$

How to solve First-Order Linear Equations

In general, say you are given

$$a(t)y' + b(t)y = c(t)$$

Then

① Divide everything by $a(t)$, assuming $a(t) \neq 0$. So

$$y' + \frac{b(t)}{a(t)}y = \frac{c(t)}{a(t)}$$

i.e. We are getting the equation into Standard Form.

② Determine $P(t)$ and $Q(t)$

$$P(t) = \frac{b(t)}{a(t)} \quad Q(t) = \frac{c(t)}{a(t)}$$

③ Find integrating factor, i.e. $u(t) = \exp\left[\int P(t)dt\right]$

④ Plug $u(t)$ and $Q(t)$ in

$$y \cdot u(t) = \int Q(t)u(t)dt$$

⑤ Integrate the RHS of ④

⑥ Divide both sides of equation from ⑤ by $u(t)$.

In today's lecture, Step ① will be done.

Example 1: Find the general solution of

① $\frac{dy}{dx} + 11y = 5$

Since the ode in standard form, ① ✓

② $P(x) = 11 \quad Q(x) = 5$

③ $u(x) = \exp\left[\int 11dx\right] = \exp[11x] = e^{11x}$

④ $y \cdot u(x) = \int Q(x)u(x)dx$
 $y \cdot e^{11x} = \int 5e^{11x}dx$

$$\textcircled{5} \quad y e^{11x} = \int 5 e^{11x} dx$$

$$y e^{11x} = \frac{5}{11} e^{11x} + C$$

$$\textcircled{6} \quad y = \frac{5}{11} \frac{e^{11x}}{e^{11x}} + \frac{C}{e^{11x}}$$

$$y = \frac{5}{11} + C e^{-11x}$$

$$\textcircled{b} \quad \frac{dy}{dx} + \left(\frac{2}{x}\right) y = 3x - 5$$

Since the ode in standard form, ① ✓

$$\textcircled{2} \quad P(x) = \frac{2}{x} \quad Q(x) = 3x - 5$$

$$\textcircled{3} \quad u(x) = \exp\left[\int P(x) dx\right] = \exp\left[\int \frac{2}{x} dx\right]$$

$$= \exp[2 \ln x] = \exp[\ln x^2] = x^2$$

$$\textcircled{4} \quad y \cdot u(x) = \int Q(x) u(x) dx$$

$$y \cdot x^2 = \int (3x - 5) x^2 dx$$

$$\textcircled{5} \quad y \cdot x^2 = \int (3x^3 - 5x^2) dx$$

$$y \cdot x^2 = \frac{3x^4}{4} - \frac{5x^3}{3} + C$$

$$\textcircled{6} \quad y = \frac{3}{4} \frac{x^4}{x^2} - \frac{5}{3} \frac{x^3}{x^2} + \frac{C}{x^2}$$

$$= \frac{3}{4} x^2 - \frac{5}{3} x + \frac{C}{x^2}$$

$$\textcircled{c} \quad y' - y = 19$$

Since the ode in standard form, ① ✓

$$\textcircled{2} \quad P(x) = -1 \quad Q(x) = 19$$

$$\textcircled{3} u(x) = \exp\left[\int P(x) dx\right] = \exp\left[\int -1 dx\right] = \exp[-x] \\ = e^{-x}$$

$$\textcircled{4} y \cdot u(x) = \int Q(x) u(x) dx \\ y \cdot e^{-x} = \int 19e^{-x} dx$$

$$\textcircled{5} y \cdot e^{-x} = \int 19e^{-x} dx \\ y \cdot e^{-x} = -19e^{-x} + C$$

$$\textcircled{6} y = \frac{-19e^{-x}}{e^{-x}} + \frac{C}{e^{-x}} \\ = -19 + Ce^x$$

$$\textcircled{d} \frac{dy}{dx} + 9y = 3e^{-9x}$$

Since the ode in standard form, $\textcircled{1} \checkmark$

$$\textcircled{2} P(x) = 9 \quad Q(x) = 3e^{-9x}$$

$$\textcircled{3} u(x) = \exp\left[\int P(x) dx\right] = \exp\left[\int 9 dx\right] = \exp[9x] = e^{9x}$$

$$\textcircled{4} y \cdot u(x) = \int Q(x) u(x) dx \\ y \cdot e^{9x} = \int 3e^{-9x} e^{9x} dx \\ y \cdot e^{9x} = \int 3 dx$$

$$\textcircled{5} y \cdot e^{9x} = \int 3 dx \\ y \cdot e^{9x} = 3x + C$$

$$\textcircled{6} y = (3x + C)e^{-9x}$$