Lesson 22: First-Order Linear Differential Equations II

Recall from Last Time, given an equation of the form

$$y' + P(t)y = Q(t)$$

a solution is given by

$$y \cdot u(t) = \int Q(t)\,u(t)\,dt$$

where $u(t) = \exp\left[\int P(t)\,dt\right]$

How to solve First-Order Linear Equations

① Using simply algebra, rewrite your equation to be

$$y' + P(t)y = Q(t)$$

i.e. We are getting the equation into Standard Form.

② Determine $P(t)$ and $Q(t)$

③ Find integrating factor. i.e. $u(t) = \exp\left[\int P(t)\,dt\right]$

④ Plug $u(t)$ and $Q(t)$ in $y \cdot u(t) = \int Q(t)\,u(t)\,dt$

⑤ Integrate the RHS of ④.

⑥ Divide both sides of the equation from ⑤ by $u(t)$.

Example 1: Find the general solution of the following equations:

ⓐ $t\dfrac{dy}{dt} + 12y = \dfrac{1}{t}$

① $\dfrac{t}{t}\dfrac{dy}{dt} + \dfrac{12}{t}y = \dfrac{1}{t}\cdot\dfrac{1}{t}$

$$\frac{dy}{dt} + \frac{12}{t}y = \frac{1}{t^2}$$

② $P(t) = \dfrac{12}{t}$ $\quad Q(t) = \dfrac{1}{t^2}$

③ $u(t) = \exp\left[\int P(t)\,dt\right] = \exp\left[\int \frac{12}{t}\,dt\right]$

$= \exp\left[12\ln(t)\right] = \exp\left[\ln(t^{12})\right] = t^{12}$

④ $y \cdot u(t) = \int Q(t)\,u(t)\,dt$

$y \cdot t^{12} = \int \frac{1}{t^2} \cdot t^{12}\,dt$

$y \cdot t^{12} = \int t^{10}\,dt$

⑤ $y \cdot t^{12} = \frac{t^{11}}{11} + C$

⑥ $y = \frac{1}{t^{12}} \cdot \frac{t^{11}}{11} + \frac{C}{t^{12}}$

$= \frac{1}{11t} + \frac{C}{t^{12}}$

⑥ $2y' + 10y = 20e^{-10x}$

① $\frac{2y'}{2} + \frac{10y}{2} = \frac{20e^{-10x}}{2}$

$y' + 5y = 10e^{-10x}$

② $P(x) = 5 \qquad Q(x) = 10e^{-10x}$

③ $u(x) = \exp\left[\int P(x)\,dx\right] = \exp\left[\int 5\,dx\right] = e^{5x}$

④ $y \cdot u(x) = \int Q(x)\,u(x)\,dx$

$y \cdot e^{5x} = \int 10e^{-10x}e^{5x}\,dx$

$y \cdot e^{5x} = \int 10e^{-5x}\,dx$

⑤ $y \cdot e^{5x} = \frac{10}{-5} e^{-5x} + C$

$y \cdot e^{5x} = -2e^{-5x} + C$

⑥ $y = -\frac{2e^{-5x}}{e^{+5x}} + \frac{C}{e^{5x}}$

$y = -2e^{-10x} + Ce^{-5x}$

ⓒ $(x-4)y' + y = x^2+1$

① $\frac{(x-4)y'}{x-4} + \frac{1}{x-4}y = \frac{x^2+1}{x-4}$

$y' + \frac{1}{x-4}y = \frac{x^2+1}{x-4}$

② $P(x) = \frac{1}{x-4}$ $\quad Q(x) = \frac{x^2+1}{x-4}$

③ $u(x) = \exp\left[\int P(x)\,dx\right] = \exp\left[\int \frac{1}{x-4}\,dx\right]$

$= \exp[\ln(x-4)] = x-4$

④ $y \cdot u(x) = \int Q(x)u(x)\,dx$

$y \cdot (x-4) = \int \frac{x^2+1}{x-4} \cdot (x-4)\,dx$

$y \cdot (x-4) = \int (x^2+1)\,dx$

⑤ $y \cdot (x-4) = \frac{x^3}{3} + x + C$

⑥ $y = \frac{x^3}{3(x-4)} + \frac{x}{x-4} + \frac{C}{x-4}$

ⓓ $3x^2y + x^3y' = 7\sec^2x\tan x$

① $x^3y' + 3x^2y = 7\sec^2x\tan x$

$$\frac{x^3y'}{x^3} + \frac{3x^2y}{x^3} = \frac{7\sec^2x\tan x}{x^3}$$

$$y' + \frac{3}{x}y = \frac{7\sec^2x\tan x}{x^3}$$

② $P(x) = \dfrac{3}{x}$ $\qquad Q(x) = \dfrac{7\sec^2x\tan x}{x^3}$

③ $u(x) = \exp\left[\int P(x)\,dx\right] = \exp\left[\int \frac{3}{x}\,dx\right] = \exp\left[3\ln(x)\right]$

$$= \exp\left[\ln(x^3)\right] = x^3$$

④ $y \cdot u(x) = \int Q(x)u(x)\,dx$

$$y \cdot x^3 = \int \frac{7\sec^2x\tan x}{x^3} \cdot x^3\,dx$$

$$y \cdot x^3 = \int 7\sec^2x\tan x\,dx$$

⑤ $u = \tan x \quad dx = \sec^2x\,dx$

$$y \cdot x^3 = \int 7u\,du$$

$$y \cdot x^3 = \frac{7u^2}{2} + C$$

$$y \cdot x^3 = \frac{7}{2}\tan^2x + C$$

⑥ $y = \dfrac{7}{2}\dfrac{\tan^2x}{x^3} + \dfrac{C}{x^3}$

ⓔ $x^4y' + x^3y = 4x^4\cos(x)$

① $\dfrac{x^4y'}{x^4} + \dfrac{x^3}{x^4}y = \dfrac{4x^4}{x^4}\cos(x)$

$$y' + \frac{1}{x}y = 4\cos(x)$$

② $P(x) = \frac{1}{x}$ $Q(x) = 4\cos(x)$

③ $u(x) = \exp[\int P(x)dx] = \exp\left[\int \frac{1}{x}dx\right] = \exp[\ln x] = x$

④ $y \cdot u(x) = \int Q(x)u(x)dx$
$y \cdot x = \int 4\cos(x) \cdot x\, dx$

⑤ $u = x$ $dv = 4\cos(x)\,dx$
$du = dx$ $v = 4\sin x$

$y \cdot x = uv - \int v\,du$
$y \cdot x = 4x\sin x - \int 4\sin x\, dx$

$y \cdot x = 4x\sin x + 4\cos x + C$
$y = \frac{4x\sin x}{x} + \frac{4\cos x}{x} + \frac{C}{x}$
$y = 4\sin x + \frac{4\cos x}{x} + \frac{C}{x}$