Lesson 23: Geometric Series & Convergence

Definition: Given an infinite sequence $a_0, a_1, a_2, \ldots$ of #s if we add all of the numbers in the sequence together we have an infinite series.

$$\sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + \ldots$$

Definition: If we look at just the first n terms in the series, this is called the n-th partial sum.

$$S_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \ldots + a_n$$

$$S_n = \sum_{k=0}^{n-1} a_k = a_0 + a_1 + \ldots + a_{n-1}$$

Note that in either case, we are adding up the first n terms of the series. The only difference is the indexing.

The series is said to be convergent if $\lim_{n \to \infty} S_n$ exists and is equal to a finite real number. If $\lim_{n \to \infty} S_n$ is infinite or does not exist, then the series is said to be divergent.

Example 1: Find the fourth partial sum of the series of $\sum_{n=1}^{\infty} n^2$

Remember we want to find the first four terms and sum them.

$$S_4 = 1^2 + 2^2 + 3^2 + 4^2 = 30$$

Example 2: Use summation notation to write the series in compact form.

ⓐ $e + \frac{e^2}{2} + \frac{e^3}{6} + \frac{e^4}{24} + \frac{e^5}{120} + \ldots$

First, look at the numerators, we see $e, e^2, e^3, e^4, e^5, \ldots$ If we start the series at $n=1$, we can see a_n will have e^n in the numerator.

Now the denominator is tricky. So let's find a pattern

$2 \rightarrow 6$ multiply by 3

$6 \rightarrow 24$ multiply by 4

$24 \rightarrow 120$ multiply by 5

$\vdots$ $\vdots$

multiply by n

Overall, we can say the series is

$$\sum_{n=1}^{\infty} \frac{e^n}{2 \cdot 3 \cdot 4 \cdot \ldots \cdot n}$$

Note we can rewrite the bottom using factorial

Recall $n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n$

So $\sum_{n=1}^{\infty} \frac{e^n}{n!}$

(b) $-3 + \frac{9}{4} - \frac{27}{9} + \frac{81}{16} - \ldots$

First notice that each term alternates from $-$ and $+$. So a_n will have $(-1)^n$ term, assuming we start the sum with $n=1$.

Secondly, the numerators are just powers of 3. So a_n's numerator will have 3^n.

Lastly, the denominators are perfect squares. So a_n's denominator will have n^2.

Put all those points together and we get

$$\sum_{n=1}^{\infty} \frac{(-1)^n 3^n}{n^2} = \sum_{n=1}^{\infty} \frac{(-3)^n}{n^2}$$

Example 3: Use summation notation to write the series in compact form.

$$0.\overline{2}$$

Note $0.\overline{2} = 0.2222\ldots = \frac{2}{10} + \frac{2}{100} + \frac{2}{1000} + \frac{2}{10000} + \ldots$

$$= \frac{2}{10}\left(1 + \frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \ldots\right)$$

$$= \frac{2}{10}\left(1 + \frac{1}{10} + \frac{1}{10^2} + \frac{1}{10^3} + \ldots\right)$$

$$= \frac{2}{10}\sum_{n=0}^{\infty}\left(\frac{1}{10}\right)^n$$

Geometric Series

If $0 < |r| < 1$, then $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$ is a geometric sum

Example 4: Compute

ⓐ $\sum_{n=1}^{\infty}\left(\frac{9}{22}\right)^n$

First we see that n starts at 1 not 0. So let's add and subtract $a_0 = \left(\frac{9}{22}\right)^0 = 1$ term

$$\sum_{n=1}^{\infty}\left(\frac{9}{22}\right)^n = -1 + \left(\frac{9}{22}\right)^0 + \sum_{n=1}^{\infty}\left(\frac{9}{22}\right)^n = -1 + \sum_{n=0}^{\infty}\left(\frac{9}{22}\right)^n$$

Then check if your r (the # under the power of n) is indeed $0 < |r| < 1$. Which it does, so we can use the formula

$$\sum_{n=0}^{\infty}\left(\frac{9}{22}\right)^n = \frac{1}{1 - 9/22} = \frac{1}{13/22} = \frac{22}{13}$$

Hence $\sum_{n=1}^{\infty}\left(\frac{9}{22}\right)^n = -1 + \frac{22}{13} = \frac{9}{13}$

ⓑ $\sum_{n=0}^{\infty}\left(\frac{3}{2}\right)^n$

We see $n=0$. So we immediately jump to checking if $r = \frac{3}{2}$

is indeed $0<|r|<1$. Which it doesn't $\Rightarrow$ diverges

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{4^{n+2}}$

First we see that n starts at 1 not 0. So let's add and subtract $a_0 = \frac{3^0}{4^{0+2}} = \frac{1}{4^2} = \frac{1}{16}$ term.

$$\sum_{n=1}^{\infty} \frac{3^n}{4^{n+2}} = -\frac{1}{16} + \frac{3^0}{4^{0+2}} + \sum_{n=1}^{\infty} \frac{3^n}{4^{n+2}} = -\frac{1}{16} + \sum_{n=0}^{\infty} \frac{3^n}{4^{n+2}}$$

Next, we want to get $(\ )^n$.

$$\sum_{n=1}^{\infty} \frac{3^n}{4^{n+2}} = -\frac{1}{16} + \sum_{n=0}^{\infty} \frac{3^n}{4^n \cdot 4^2} = -\frac{1}{16} + \sum_{n=0}^{\infty} \frac{1}{16}\left(\frac{3}{4}\right)^n$$

Check if $r = \frac{3}{4}$ is between $0<|r|<1$, which it is. So apply the formula

$$\sum_{n=0}^{\infty} \frac{1}{16}\left(\frac{3}{4}\right)^n = \frac{1/16}{1-3/4} = \frac{1/16}{1/4} = \frac{1}{16} \cdot \frac{4}{1} = \frac{1}{4}$$

Hence $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{4^{n+2}} = -\frac{1}{16} + \frac{1}{4} = \frac{3}{16}$

(d) $\displaystyle\sum_{n=0}^{\infty} \left(\frac{3}{7^n} + \frac{4}{5^n}\right)$

First, let's rewrite the sum.

$$\sum_{n=0}^{\infty} \left(\frac{3}{7^n} + \frac{4}{5^n}\right) = \sum_{n=0}^{\infty} 3\left(\frac{1}{7}\right)^n + \sum_{n=0}^{\infty} 4\left(\frac{1}{5}\right)^n$$

Now check if both $(\ )^n$ terms satisfy $0<|r|<1$. So $r = 1/7$ is between 0 and 1, and so is $r = 1/5$. Hence we can use the formulas on each.

$$\sum_{n=0}^{\infty} 3\left(\frac{1}{7}\right)^n = \frac{3}{1-1/7} = \frac{3}{6/7} = 3 \cdot \frac{7}{6} = \frac{7}{2}$$

$$\sum_{n=0}^{\infty} 4\left(\frac{1}{5}\right)^n = \frac{4}{1-1/5} = \frac{4}{4/5} = 4 \cdot \frac{5}{4} = 5$$

Hence $\displaystyle\sum_{n=0}^{\infty} \left(\frac{3}{7^n} + \frac{4}{5^n}\right) = \frac{7}{2} + 5 = \frac{17}{2}$

(e) $\sum_{n=0}^{\infty} 10e^{-0.6n}$

First, let's rewrite the sum

$$\sum_{n=0}^{\infty} 10e^{-0.6n} = \sum_{n=0}^{\infty} 10(e^{-0.6})^n$$

Check if $r = e^{-0.6}$ is between $0<|r|<1$, which it is.

So apply the formula.

$$\sum_{n=0}^{\infty} 10(e^{-0.6})^n = \frac{10}{1-e^{-0.6}}$$

Hence $\sum_{n=0}^{\infty} 10e^{-0.6n} = \frac{10}{1-e^{-0.6}}$

(f) $\sum_{n=0}^{\infty} \frac{6(-1)^n}{3^{2n}}$

First, let's rewrite the sum

$$\sum_{n=0}^{\infty} \frac{6(-1)^n}{3^{2n}} = \sum_{n=0}^{\infty} \frac{6(-1)^n}{(3^2)^n} = \sum_{n=0}^{\infty} 6\left(\frac{-1}{3^2}\right)^n = \sum_{n=0}^{\infty} 6\left(\frac{-1}{9}\right)^n$$

Check if $r = -1/9$ is between $0<|r|<1$, which it is.

So apply the formula.

$$\sum_{n=0}^{\infty} 6\left(\frac{-1}{9}\right)^n = \frac{6}{1-(-1/9)} = \frac{6}{10/9} = \frac{27}{5}$$

Hence $\sum_{n=0}^{\infty} \frac{6(-1)^n}{3^{2n}} = \frac{27}{5}$