

# Lesson 28: Higher Order Partial Derivatives

Recall from last class, the partial derivatives of  $z = f(x, y)$  are

$$f_x = \frac{df}{dx}$$

$$f_y = \frac{df}{dy}$$

Higher Order Partial Derivatives are similar to Higher Order Derivatives (Lesson 13) from Applied Calculus I. The main difference is that we can mix partials (i.e. first do  $x$  then  $y$  or vice versa).

Notation/Definition: Taking the partial derivatives of the partials  $f_x$  and  $f_y$ , we have the following second-order partials.

$$\textcircled{1} \frac{d}{dx} (f_x) = \frac{d}{dx} \left( \frac{df}{dx} \right) = \frac{d^2 f}{dx^2} = f_{xx}$$

$$\textcircled{2} \frac{d}{dy} (f_x) = \frac{d}{dy} \left( \frac{df}{dx} \right) = \frac{d^2 f}{dy dx} = f_{xy}$$

$$\textcircled{3} \frac{d}{dx} (f_y) = \frac{d}{dx} \left( \frac{df}{dy} \right) = \frac{d^2 f}{dx dy} = f_{yx}$$

$$\textcircled{4} \frac{d}{dy} (f_y) = \frac{d}{dy} \left( \frac{df}{dy} \right) = \frac{d^2 f}{dy^2} = f_{yy}$$

Fact: While not true in general, but for the purpose of this class, we have

$$f_{xy} = f_{yx}$$

So  $\textcircled{2}$  and  $\textcircled{3}$ , in the definition above, can be combined

Example 1: Compute the second order partial derivatives

$$\textcircled{a} f(x, y) = 5x^2y + 2xy^3 + 3y^2$$

First find  $f_x$  and  $f_y$ .

$$f_x = 10xy + 2y^3$$

$$f_y = 5x^2 + 6xy^2 + 6y$$

Next find  $f_{xx}$ ,  $f_{xy}$ , and  $f_{yy}$  via the definition.

$$f_{xx} = \frac{d}{dx} (f_x) = \frac{d}{dx} (10xy + 2y^3) = 10y + 0 = 10y$$

$$f_{xy} = \frac{d}{dy}(f_x) = \frac{d}{dy}(10xy + 2y^3) = 10x + 6y^2$$

$$f_{yy} = \frac{d}{dy}(f_y) = \frac{d}{dy}(5x^2 + 6xy^2 + 6y) = 0 + 12xy + 6 = 12xy + 6$$

$$\textcircled{b} f(x, y) = x^2 y e^{7x}$$

First find  $f_x$  and  $f_y$ .

$$f(x, y) = y(x^2 e^{7x})$$

$$f_x = y(2xe^{7x} + x^2 e^{7x} \cdot 7) = ye^{7x}(2x + 7x^2)$$

Product  
Rule

$$f(x, y) = (x^2 e^{7x}) y$$

$$f_y = x^2 e^{7x}$$

Next find  $f_{xx}$ ,  $f_{xy}$ , and  $f_{yy}$  via the definition.

$$f_{xx} = \frac{d}{dx}(f_x) = \frac{d}{dx}(ye^{7x}(2x + 7x^2)) = y \frac{d}{dx}(e^{7x}(2x + 7x^2))$$

$$= y(7e^{7x}(2x + 7x^2) + e^{7x}(2 + 14x))$$

$$= ye^{7x}[7(2x + 7x^2) + (2 + 14x)] = ye^{7x}[14x + 49x^2 + 2 + 14x]$$

$$= ye^{7x}(49x^2 + 28x + 2)$$

$$f_{xy} = \frac{d}{dy}(f_y) = \frac{d}{dy}(ye^{7x}(2x + 7x^2)) = e^{7x}(2x + 7x^2) \frac{d}{dy}(y) = e^{7x}(2x + 7x^2)$$

$$f_{yy} = \frac{d}{dy}(x^2 e^{7x}) = 0 \quad \text{b/c there is no } y \text{ terms}$$

$$\textcircled{c} f(x, y) = \ln(6x^2 + 7y)$$

→ First find  $f_x$  and  $f_y$ .

$$f_x = \frac{1}{6x^2 + 7y} \cdot \frac{d}{dx}(6x^2 + 7y) = \frac{12x}{6x^2 + 7y}$$

$$f_y = \frac{1}{6x^2 + 7y} \cdot \frac{d}{dy}(6x^2 + 7y) = \frac{7}{6x^2 + 7y}$$

Chain  
Rule

Next find  $f_{xx}$ ,  $f_{xy}$ , and  $f_{yy}$  via the definition.

$$f_{xx} = \frac{d}{dx}(f_x) = \frac{d}{dx}\left(\frac{12x}{6x^2+7y}\right) = \frac{12(6x^2+7y) - 12x(12x+0)}{(6x^2+7y)^2}$$
$$= \frac{72x^2 + 84y - 144x^2}{(6x^2+7y)^2}$$

$$f_x = 12x(6x^2+7y)^{-1}$$

$$f_{xy} = \frac{d}{dy}(f_x) = 12x \cdot (-1)(6x^2+7y)^{-2} \cdot (0+7) = \frac{-84x}{(6x^2+7y)^2}$$

$$f_y = 7(6x^2+7y)^{-1}$$

$$f_{yy} = \frac{d}{dy}(f_y) = 7 \cdot (-1)(6x^2+7y)^{-2} \cdot (0+7) = \frac{-49}{(6x^2+7y)^2}$$