Lesson 33: Double Integrals I

Let $z = f(x,y)$ be a function of two variables. Similar to taking partial derivatives with respect to x and y, we can take

- $\displaystyle\int_{x=a}^{x=b} f(x,y)\,dx$ – Integrate with respect to x and Treat y as a constant
- $\displaystyle\int_{y=c}^{y=d} f(x,y)\,dy$ – Integrate with respect to y and Treat x as a constant

Combining the above integrals, we obtain the following double integrals:

- $\displaystyle\int_{y=c}^{y=d}\int_{x=a}^{x=b} f(x,y)\,dx\,dy = \int_{y=c}^{y=d}\left(\int_{x=a}^{x=b} f(x,y)\,dx\right)dy$
- $\displaystyle\int_{x=a}^{x=b}\int_{y=c}^{y=d} f(x,y)\,dy\,dx = \int_{x=a}^{x=b}\left(\int_{y=c}^{y=d} f(x,y)\,dy\right)dx$

<u>Example 1</u>: Evaluate

ⓐ $\displaystyle\int_0^5\int_0^3 (x+y)\,dy\,dx = \int_{x=0}^{x=5}\left(\int_{y=0}^{y=3}(x+y)\,dy\right)dx$

$$= \int_{x=0}^{x=5}\left(\left(xy + \frac{y^2}{2}\right)\Big]_{y=0}^{y=3}\right)dx$$

$$= \int_{x=0}^{x=5}\left(3x + \frac{9}{2} - \left(0\cdot x - \frac{0^2}{2}\right)\right)dx$$

$$= \int_{x=0}^{x=5}\left(3x + \frac{9}{2}\right)dx$$

$$= \left(\frac{3x^2}{2} + \frac{9}{2}x\right)\Big]_{x=0}^{x=5}$$

$$= \frac{3(5)^2}{2} + \frac{9}{2}(5) - \left(\frac{3(0)^2}{2} + \frac{9}{2}(0)\right)$$

$$= 60$$

ⓑ $\int_1^2 \int_0^1 x^2 y \, dy \, dx = \int_{x=1}^{x=2} \left(\int_{y=0}^{y=1} x^2 y \, dy \right) dx$

$= \int_{x=1}^{x=2} x^2 \cdot \left(\int_{y=0}^{y=1} y \, dy \right) dx$

$= \int_{x=1}^{x=2} x^2 \left(\frac{y^2}{2} \right) \Big]_{y=0}^{y=1} dx$

$= \int_{x=1}^{x=2} x^2 \left(\frac{1^2}{2} - \frac{0^2}{2} \right) dx$

$= \int_{x=1}^{x=2} \frac{1}{2} x^2 \, dx$

$= \frac{1}{2} \cdot \frac{x^3}{3} \Big]_{x=1}^{x=2}$

$= \frac{2^3}{6} - \frac{1^3}{6}$

$= \frac{7}{6}$

ⓒ $\int_0^{\pi/2} \int_0^1 16y^4 \cos(x) \, dy \, dx = \int_{x=0}^{x=\pi/2} \left(\int_{y=0}^{y=1} 16y^4 \cos(x) \, dy \right) dx$

$= \int_{x=0}^{x=\pi/2} \cos(x) \cdot \left(\int_{y=0}^{y=1} 16y^4 \, dy \right) dx$

$= \int_{x=0}^{x=\pi/2} \cos(x) \cdot \left(\frac{16y^5}{5} \right) \Big]_{y=0}^{y=1} dx$

$= \int_{x=0}^{x=\pi/2} \cos(x) \left(\frac{16(1)^5}{5} - \frac{16(0)^5}{5} \right) dx$

$= \int_{x=0}^{x=\pi/2} \frac{16}{5} \cos(x) \, dx$

$= \frac{16}{5} \sin(x) \Big]_{x=0}^{x=\pi/2}$

$= \frac{16}{5} \left(\sin\left(\frac{\pi}{2}\right) - \sin(0) \right)$

$= \frac{16}{5} (1 - 0)$

$= 16/5$

(d) $\int_{\pi/6}^{\pi/2}\int_{-1}^{2} \cos(y)\,dx\,dy = \int_{y=\pi/6}^{y=\pi/2}\left(\int_{x=-1}^{x=2} \cos(y)\,dx\right)dy$

$= \int_{y=\pi/6}^{y=\pi/2} \cos(y)\cdot\left(\int_{x=-1}^{x=2} dx\right)dy$

$= \int_{y=\pi/6}^{y=\pi/2} \cos(y)\cdot \left. x \right]_{x=-1}^{x=2} dy$

$= \int_{y=\pi/6}^{y=\pi/2} \cos(y)\cdot(2-(-1))\,dy$

$= \int_{y=\pi/6}^{y=\pi/2} 3\cos(y)\,dy$

$= \left. 3\sin y \right]_{y=\pi/6}^{y=\pi/2}$

$= 3\left[\sin\left(\frac{\pi}{2}\right) - \sin\left(\frac{\pi}{6}\right)\right]$

$= 3\left[1-\frac{1}{2}\right]$

$= \frac{3}{2}$

(e) $\int_0^5\int_0^3 x^2 e^{2y}\,dx\,dy = \int_{y=0}^{y=5}\int_{x=0}^{x=3} x^2 e^{2y}\,dx\,dy$

$= \int_{y=0}^{y=5} e^{2y}\cdot\left(\int_{x=0}^{x=3} x^2\,dx\right)dy$

$= \int_{y=0}^{y=5} e^{2y}\left.\left(\frac{x^3}{3}\right)\right]_{x=0}^{x=3} dy$

$= \int_{y=0}^{y=5} e^{2y}\left(\frac{3^3}{3}-\frac{0^3}{3}\right)dy$

$= \int_{y=0}^{y=5} 9e^{2y}\,dy$

$= \left.\frac{9}{2}e^{2y}\right]_0^5$

$= \frac{9}{2}\left(e^{10}-e^{0}\right) = \frac{9}{2}\left(e^{10}-1\right)$