Lesson 34: Double Integrals II

Let's recall how to integrate double integrals with the following integrals:

Example 1: Evaluate

ⓐ $\int_0^7 \int_0^y 5xy\,dx\,dy = \int_{y=0}^{y=7} \left(\int_{x=0}^{x=y} 5xy\,dx \right) dy$

$= \int_{y=0}^{y=7} 5y \cdot \left(\int_{x=0}^{x=y} x\,dx \right) dy$

$= \int_{y=0}^{y=7} 5y \left(\frac{x^2}{2} \right) \Big]_{x=0}^{x=y} dy$

$= \int_{y=0}^{y=7} 5y \left(\frac{y^2}{2} - \frac{0^2}{2} \right) dy$

$= \int_{y=0}^{y=7} 5y \cdot \frac{y^2}{2}\,dy$

$= \frac{5}{2} \int_{y=0}^{y=7} y^3\,dy$

$= \frac{5}{2} \cdot \frac{y^4}{4} \Big]_0^7$

$= \frac{5}{8} \left(7^4 - 0^4 \right) = \frac{12005}{8}$

ⓑ $\int_3^4 \int_3^x \frac{6x}{y^2}\,dy\,dx = \int_{x=3}^{x=4} \left(\int_{y=3}^{y=x} 6xy^{-2}\,dy \right) dx$

$= \int_{x=3}^{x=4} 6x \cdot \left(\int_{y=3}^{y=x} y^{-2}\,dy \right) dx$

$= \int_{x=3}^{x=4} 6x \cdot \left(-y^{-1} \right) \Big]_{y=3}^{y=x} dx$

$= \int_{x=3}^{x=4} 6x \left(\frac{-1}{y} \right) \Big]_{y=3}^{y=x} dx$

$= \int_{x=3}^{x=4} 6x \left(-\frac{1}{x} + \frac{1}{3} \right) dx$

Last Class, we introduced

$$\int_{y=c}^{y=d}\left(\int_{x=a}^{x=b} f(x,y)\,dx\right)dy \quad \text{and} \quad \int_{x=a}^{x=b}\left(\int_{y=c}^{y=d} f(x,y)\,dy\right)dx$$

A geometric interpertation of these double integrals is we are finding the volume below $z = f(x,y)$ above the region

$$R = \{(x,y) \mid a \le x \le b \,;\; c \le y \le d\}$$

We can denote the integrals above by just one

$$\iint_R f(x,y)\,dA$$

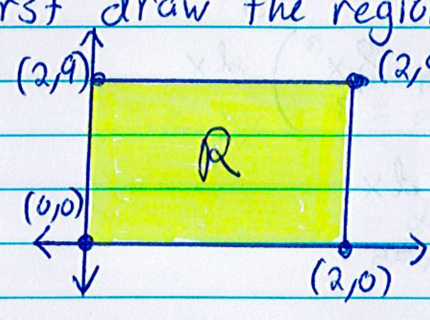
where R is the **domain of integration.**

To solve the integrals of the form $\iint_R f(x,y)\,dA$, we start by drawing the region. We do this to determine the bounds of our integrals.

Example 1: Evaluate the integral $\iint_R 10x^3y\,dA$ where R is the rectangle with vertices $(0,0)$, $(2,0)$, $(0,9)$, and $(2,9)$.

First draw the region, R. We can see that $0 \le x \le 2$ and $0 \le y \le 9$. So

$$R = \{(x,y) \mid 0 \le x \le 2,\ 0 \le y \le 9\}$$



Hence $\iint_R 10x^3y\,dA = \int_{x=0}^{x=2}\left(\int_{y=0}^{y=9} 10x^3 \cdot y\,dy\right)dx$

$$= \int_{x=0}^{x=2}\left(10x^3 \cdot \frac{y^2}{2}\Big]_{y=0}^{y=9}\right)dx$$

$$= \int_{x=0}^{x=2}\left(10x^3\left(\frac{9^2}{2} - \frac{0^2}{2}\right)\right)dx$$

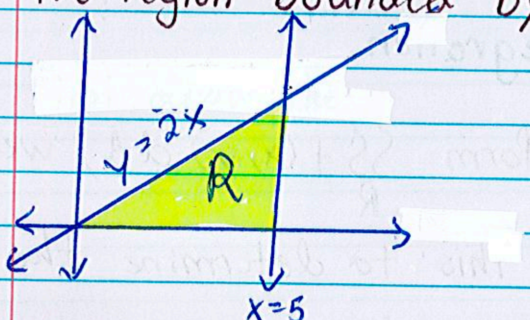
$$= \int_{x=0}^{x=2} \left(10x^3 \cdot \frac{81}{2}\right) dx$$

$$= \int_{x=0}^{x=2} 405x^3\, dx$$

$$= \frac{405}{4} x^4 \Big]_{x=0}^{x=2}$$

$$= \frac{405}{4}(2^4 - 0^4) = 1620$$

Example 2: Evaluate the integral $\iint_R (x^2+y^2)\, dA$ where R is the region bounded by the lines $y=2x$, $x=5$, and the x-axis.



First draw the region, R. We can see that

$$R = \{(x,y) \mid 0 \le x \le 5,\ 0 \le y \le 2x\}$$

Hence $$\iint_R (x^2+y^2)\, dA = \int_{x=0}^{x=5}\left(\int_{y=0}^{y=2x} (x^2+y^2)\, dy\right) dx$$

$$= \int_{x=0}^{x=5}\left(\left(x^2 \cdot y + \frac{y^3}{3}\right)\Big]_{y=0}^{y=2x}\right) dx$$

$$= \int_{x=0}^{x=5}\left(x^2(2x) + \frac{(2x)^3}{3} - \left(x^2 \cdot 0 + \frac{0^3}{3}\right)\right) dx$$

$$= \int_{x=0}^{x=5}\left(2x^3 + \frac{8x^3}{3}\right) dx$$

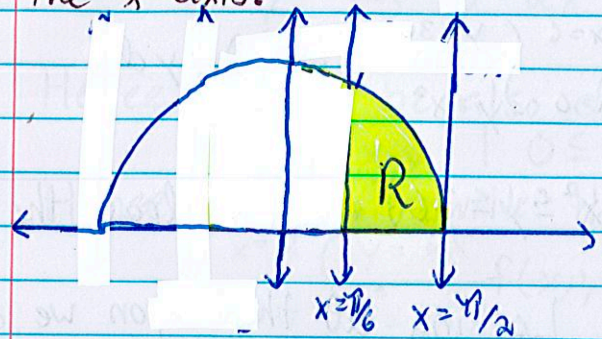
$$= \int_{x=0}^{x=5} \frac{14}{3} x^3\, dx$$

$$= \frac{14}{3} \cdot \frac{x^4}{4}\Big]_{x=0}^{x=5}$$

$$= \frac{7}{6} x^4 \Big]_{x=0}^{x=5}$$

$$= \frac{7}{6}(5^4 - 0^4) = \frac{4375}{6}$$

Example 3: Evaluate the integral $\iint_R 6\sin^2(x)\,dA$ where R is the region bounded by the curves $y = \cos(x)$, $x = \pi/6$, $x = \pi/2$ and the x-axis.



First draw the region, R. We can see that

$$R = \left\{ (x,y) \;\middle|\; \frac{\pi}{6} \le x \le \frac{\pi}{2},\ 0 \le y \le \cos x \right\}$$

Hence
$$\iint_R 6\sin^2(x)\,dA = \int_{x=\pi/6}^{x=\pi/2}\left(\int_{y=0}^{y=\cos(x)} 6(\sin x)^2\,dy\right)dx$$
$$= \int_{x=\pi/6}^{x=\pi/2}\left(6(\sin x)^2 \cdot y\Big]_{y=0}^{y=\cos x}\right)dx$$
$$= \int_{x=\pi/6}^{x=\pi/2}\left(6(\sin x)^2 \cdot (\cos x - 0)\right)dx$$
$$= \int_{x=\pi/6}^{x=\pi/2} 6(\sin x)^2 \cdot \cos x\,dx$$

$$\frac{u = \sin x}{du = \cos x\,dx} \quad \int 6u^2\,du$$
$$= \frac{6u^3}{3} = 2u^3$$
$$= 2(\sin x)^3\Big]_{x=\pi/6}^{x=\pi/2}$$
$$= 2\left(\left(\sin\left(\frac{\pi}{2}\right)\right)^3 - \left(\sin\left(\frac{\pi}{6}\right)\right)^3\right)$$
$$= 2\left((1)^3 - \left(\frac{1}{2}\right)^3\right) = 2\left(1 - \frac{1}{8}\right) = 2 \cdot \frac{7}{8} = \frac{7}{4}$$

Note that Examples 2 and 3 could have been done with dxdy as the order of integration.

So a good question to ask is when to use dxdy or dydx? We will answer that next time.