Lesson 35: Double Integrals

Recall from Lesson 5, the formula for average value:

For $f(x)$ defined on $[a,b]$, the average value of $f(x)$ on $[a,b]$ is

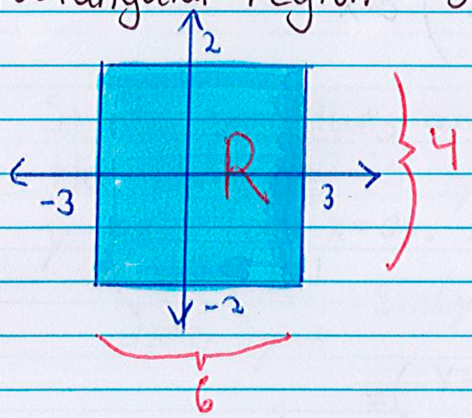
$$f_{AVE}(x) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

The multivariable average value formula follows:

For $f(x,y)$ defined on a region, R, the average value of $f(x,y)$ over the region, R, is given by

$$f_{AVE}(x,y) = \frac{1}{A}\iint_R f(x,y)\,dA$$ where A is the area of R.

Example 1: Find the average of $f(x,y) = 12 - x^2 - y^2$ in a rectangular region $-3 \le x \le 3$, $-2 \le y \le 2$.



First draw the region. With the drawing, find the area of R.

Area of R $= 6 \times 4 = 24$

Note we are given the bounds for our integral. So

$$f_{AVE}(x,y) = \frac{1}{24}\int_{x=-3}^{x=3}\int_{y=-2}^{y=2}(12-x^2-y^2)\,dy\,dx$$

Now integrate.

$$f_{AVE}(x,y) = \frac{1}{24}\int_{x=-3}^{x=3}\left(\left(12y - x^2y - \frac{y^3}{3}\right)\right]_{y=-2}^{y=2} dx$$

$$= \frac{1}{24}\int_{x=-3}^{x=3}\left(12(2) - 2x^2 - \frac{2^3}{3} - \left(12(-2) - x^2(-2) - \frac{(-2)^3}{3}\right)\right)dx$$

$$= \frac{1}{24}\int_{x=-3}^{x=3}\left(24 - 2x^2 - \frac{8}{3} + 24 - 2x^2 - \frac{8}{3}\right)dx$$

$$= \frac{1}{24}\int_{x=-3}^{x=3}\left(\frac{128}{3} - 4x^2\right)dx$$

$$= \frac{1}{24}\left(\frac{128}{3}x - \frac{4x^3}{3}\right)\Bigg]_{x=-3}^{x=3}$$

$$= \frac{1}{24}\left(\frac{128}{3}(3) - \frac{4(3)^3}{3} - \left(\frac{128}{3}(-3) - \frac{4(-3)^3}{3}\right)\right) = \frac{28}{3}$$

Recall from Last Time, given a function $z=f(x,y)$ and a region, R, in the xy-plane, we have

$$\iint_R f(x,y)\,dA$$

To solve these integrals, we start by drawing the region. We do this to determine the bounds of our integrals.

So a good question to ask is when to use dxdy or dydx?

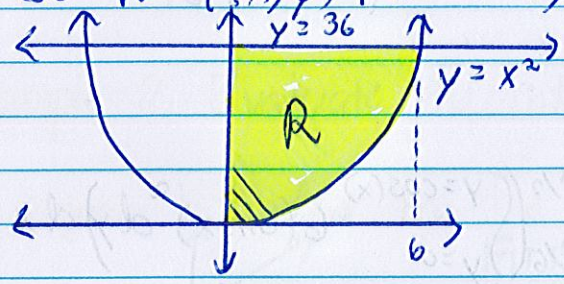
Before we answer that let's practice switching the order of integration.

To switch the order of integration, we need to draw a picture.

Example **2**: Switch the order of integration on the following integrals.

ⓐ $\int_0^6 \int_{x^2}^{36} f(x,y)\,dy\,dx = \int_{x=0}^{x=6} \int_{y=x^2}^{y=36} f(x,y)\,dy\,dx$

So $R = \{(x,y) \mid 0 \le x \le 6,\ x^2 \le y \le 36\}$. Let's draw the region.



Looking at the region we can describe R in another way. The y-values are $0 \le y \le 6$. As for x, we see that x lies between the y-axis ($x=0$) and the parabola ($y=x^2$). Let's get $y=x^2$ in terms of $x = \_\_$.

$$x = \pm\sqrt{y}$$

Since $x \ge 0$, $x = \sqrt{y}$. Hence R can be also described by

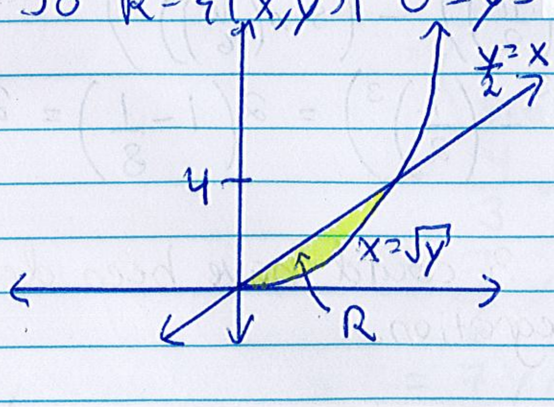
$$R = \{(x,y) \mid 0 \le y \le 6,\ 0 \le x \le \sqrt{y}\}$$

Hence we can rewrite the integral to be

$$\int_{y=0}^{y=6} \int_{x=0}^{x=\sqrt{y}} f(x,y)\,dx\,dy$$

ⓑ $\int_0^4 \int_{y/2}^{\sqrt{y}} f(x,y)\,dx\,dy = \int_{y=0}^{y=4} \int_{x=y/2}^{x=\sqrt{y}} f(x,y)\,dx\,dy$

So $R = \{(x,y) \mid 0 \le y \le 4,\ y/2 \le x \le \sqrt{y}\}$. Let's draw R.



Looking at the region we can describe R in another way. We see $x=0$ is the smallest value to find the largest plug $y=4$ into

$$x = \frac{y}{2} \text{ or } x = \sqrt{y}$$

So $x = \frac{4}{2} = 2$. Hence $0 \le x \le 2$. As for y, we see that y lies between $x = \sqrt{y}$ and $x = \frac{y}{2}$. Let's get both equations in terms of y.

$$x = \frac{y}{2} \Rightarrow y = 2x \qquad x = \sqrt{y} \Rightarrow y = x^2$$

Hence R can be also described by

$$R = \{(x,y) \mid 0 \leq x \leq 2,\ x^2 \leq y \leq 2x\}$$

Hence we can rewrite the integral to be

$$\int_{x=0}^{x=2} \int_{y=x^2}^{y=2x} f(x,y)\,dy\,dx$$

But sometimes the integral we obtain can't be integrated. Mainly because we don't know it's antiderivative. So when should you use dxdy or dydx is vital in these problems.

$\boxed{\Rightarrow}$

i.e. This is where switching the order of integration comes in. Given an integral with $dxdy$, we can switch it to $dydx$ (and vice versa) via the drawing of the region.

It's easier to see through some examples.

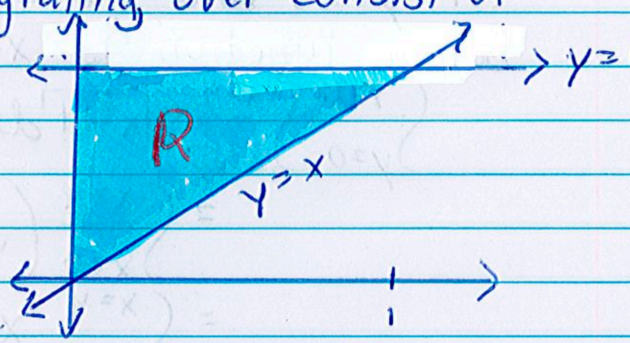
Example 3: Compute

ⓐ $\int_0^1 \int_x^1 \sin(y^2)\,dy\,dx = \int_{x=0}^{x=1} \int_{y=x}^{y=1} \sin(y^2)\,dy\,dx$

So the region we are integrating over consist of

$0 \le x \le 1,\ x \le y \le 1$

Note: $y=x$ is the bottom function, while $y=1$ is the top.



So the y-values are $0 \le y \le 1$. The x-values we see that the largest is $y=x$ (or $x=y$) and smallest is the y-axis (or $x=0$). So $0 \le x \le y$. Hence

$$\int_{x=0}^{x=1} \int_{y=x}^{y=1} \sin(y^2)\,dy\,dx = \int_{y=0}^{y=1} \int_{x=0}^{x=y} \sin(y^2)\,dx\,dy$$

$$= \int_{y=0}^{y=1} \left(\sin(y^2) \cdot x \Big]_{x=0}^{x=y} \right) dy$$

$$= \int_{y=0}^{y=1} y \sin(y^2)\,dy$$

$$\xrightarrow[\substack{du = 2y\,dy \\ du/2 = y\,dy}]{u=y^2} \int \sin(u)\,\frac{du}{2}$$

$$= -\frac{1}{2}\cos(u)$$

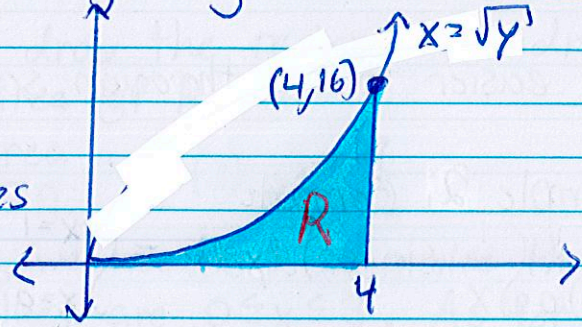
$$= -\frac{1}{2}\cos(y^2)\Big]_{y=0}^{y=1}$$

$$= -\frac{1}{2}\cos(1) + \frac{1}{2}\cos(0)$$

$$= \frac{1}{2} - \frac{1}{2}\cos(1)$$

(b) $\int_0^{16}\int_{\sqrt{y}}^{4} \sqrt{x^3+1}\,dx\,dy = \int_{y=0}^{y=16}\int_{x=\sqrt{y}}^{x=4} \sqrt{x^3+1}\,dx\,dy$

So the region we are integrating over consist of
$0 \le y \le 16,\ \sqrt{y} \le x \le 4$



So the x-values are $0 \le x \le 4$. The y-values we see the top function is $x=\sqrt{y}$ (or $y=x^2$) and the bottom function is x-axis (or $y=0$). So $0 \le y \le x^2$. Hence

$$\int_{y=0}^{y=16}\int_{x=\sqrt{y}}^{x=4} \sqrt{x^3+1}\,dx\,dy = \int_{x=0}^{x=4}\int_{y=0}^{y=x^2} \sqrt{x^3+1}\,dy\,dx$$

$$= \int_{x=0}^{x=4} \left(\sqrt{x^3+1}\cdot y \Big]_{y=0}^{y=x^2} \right) dx$$

$$= \int_{x=0}^{x=4} x^2 \sqrt{x^3+1}\,dx$$

$$\frac{u=x^3+1}{du=3x^2dx} \int u^{1/2}\,\frac{du}{3}$$
$du/3 = x^2dx$

$$= \frac{1}{3}\cdot\frac{2}{3}u^{3/2}$$

$$= \frac{2}{9}(x^3+1)^{3/2}\Big]_{x=0}^{x=4}$$

$$= \frac{2}{9}(4^3+1)^{3/2} - \frac{2}{9}(0^3+1)^{3/2}$$

$$= \frac{2}{9}\cdot(65)^{3/2} - \frac{2}{9}$$