

Lesson 3: Integration By Substitution I

Warm-Up: Determine the Inner/Outer Functions for the following:

(a) $h(x) = \sin^3 x$

Note $h(x) = (\sin x)^3$

So outer function $\Rightarrow f(x) = x^3$

inner function $\Rightarrow g(x) = \sin x$

Check $h(x) = f(g(x))$ is true.

$$\begin{aligned} f(g(x)) &= f(\sin x) \\ &= (\sin x)^3 = h(x) \end{aligned}$$

(b) $h(x) = \sqrt{3x+2}$

So outer function $\Rightarrow f(x) = \sqrt{x}$

inner function $\Rightarrow g(x) = 3x+2$

Check $h(x) = f(g(x))$ is true

$$\begin{aligned} f(g(x)) &= f(3x+2) \\ &= \sqrt{3x+2} = h(x) \end{aligned}$$

(c) $h(x) = \tan(3x^2)$

So outer function $\Rightarrow f(x) = \tan x$

inner function $\Rightarrow g(x) = 3x^2$

Check $h(x) = f(g(x))$ is true

$$\begin{aligned} f(g(x)) &= f(3x^2) \\ &= \tan(3x^2) = h(x) \end{aligned}$$

Integration By Substitution is kinda of the Integration Version of the Chain Rule. Also referred to as "Change of Variables."

Example 1: Find $\int 2x \cos(x^2) dx$.

Idea to solve Example 1 is to undo the chain rule.

First determine if you have a function within a function.

In this example,

$$\cos(x^2)$$

where $\cos(x)$ is the outer function, and
 x^2 is the inner function.

What's the derivative of the inner function?

$$2x$$

Do you see $2x$ in the integrand?

YES!!!

Let's recall chain rule: $y' = f'(g(x)) \cdot g'(x)$ for $y = f(g(x))$

So from our example: $y' = \cos(x^2) \cdot 2x$

i.e. $f(x) =$

$$g(x) = x^2$$

$$f'(x) = \cos x$$

$$g'(x) = 2x$$

So what's $f(x)$?

$$f(x) = \int f'(x) dx = \int \cos x dx = \sin x$$

Hence $y = f(g(x)) = \sin(x^2)$.

i.e. $\int 2x \cos(x^2) dx = \sin(x^2) + C$

How to fast track this method?

Do a change of variable using the inner function.

Solution to Example 1:

$$\int \underbrace{\cos(x^2)}_u \cdot \underbrace{2x dx}_{du} \quad \frac{u = x^2}{du = 2x dx} \quad \int \cos(u) du$$

$$= \sin(u) + C$$

But if we start w/ a function of x . We want to end w/ a function of x . So sub back $u=x^2$.

$$\int \cos(x^2) \cdot 2x dx = \sin(x^2) + C$$

Note we may not see what du equals in our integrand, so you may have to do some equation manipulation.

Example 2: Compute the following integrals:

$$(a) \int \sqrt{4x+1} dx \quad \frac{u=4x+1}{du=4dx} \quad \int \sqrt{u} \cdot \frac{du}{4} = \frac{1}{4} \int u^{1/2} du$$
$$\frac{du}{4} = dx$$

$$= \frac{1}{4} \cdot \frac{2}{3} u^{3/2} + C = \frac{1}{6} (4x+1)^{3/2} + C$$

$$(b) \int 4e^{2x} dx \quad \frac{u=2x}{du=2dx} \quad \int 4e^u \cdot \frac{du}{2} = 2 \int e^u du$$
$$\frac{du}{2} = dx$$

$$= 2e^u + C = 2e^{2x} + C$$

$$(c) \int 3x^5 e^{x^6} dx \quad \frac{u=x^6}{du=6x^5 dx} \quad \int 3x^5 e^u \cdot \frac{du}{6x^5} = \int \frac{1}{2} e^u du$$
$$\frac{du}{6x^5} = dx$$

$$= \frac{1}{2} e^u + C = \frac{1}{2} e^{x^6} + C$$

$$(d) \int \frac{3 \sin x}{\cos^8 x} dx = \int \frac{3 \sin x}{(\cos x)^8} dx$$

$$\frac{u=\cos x}{du=-\sin x dx} \quad \int \frac{3 \sin x}{u^8} \cdot \frac{du}{-\sin x} = -3 \int \frac{du}{u^8}$$
$$\frac{du}{-\sin x} = dx$$

$$= -3 \int u^{-8} du = -3 \frac{u^{-7}}{-7} + C = \frac{3}{7} (\cos x)^{-7} + C$$

$$\begin{aligned} \textcircled{e} \int 5 e^{\tan(14x)} \cdot \sec^2(14x) dx & \quad \begin{array}{l} u = \tan(14x) \\ du = \sec^2(14x) \cdot 14 dx \\ \frac{du}{\sec^2(14x) \cdot 14} = dx \end{array} \\ & = \int 5 e^u \sec^2(14x) \cdot \frac{du}{\sec^2(14x) \cdot 14} = \int \frac{5}{14} e^u du \\ & = \frac{5}{14} e^u + C = \frac{5}{14} e^{\tan(14x)} + C \end{aligned}$$

Example 3: Find the function $f(x)$ whose tangent line has the slope $\frac{(1+\sqrt{x})^{1/2}}{4\sqrt{x}}$ for any $x \neq 0$ and whose graph passes through

the point $(9, 5/3)$.

$$f'(x) = \frac{(1+\sqrt{x})^{1/2}}{4\sqrt{x}}$$

$$\begin{aligned} \text{So } f(x) &= \int f'(x) dx = \int \frac{(1+\sqrt{x})^{1/2}}{4\sqrt{x}} dx \quad \begin{array}{l} u = 1+\sqrt{x} = 1+x^{1/2} \\ du = \frac{1}{2} x^{-1/2} dx = \frac{dx}{2\sqrt{x}} \\ 2\sqrt{x} du = dx \end{array} \\ &= \int \frac{u^{1/2}}{4\sqrt{x}} \cdot 2\sqrt{x} du = \int \frac{1}{2} u^{1/2} du \end{aligned}$$

$$= \frac{1}{2} \cdot \frac{2}{3} u^{3/2} + C = \frac{1}{3} (1+\sqrt{x})^{3/2} + C$$

Now we need to find C . (w/ $(9, 5/3)$)

$$\frac{5}{3} = f(9) = \frac{1}{3} (1+\sqrt{9})^{3/2} + C$$

$$\frac{5}{3} = \frac{8}{3} + C$$

$$-1 = -\frac{3}{3} = C$$

$$\text{So } f(x) = \frac{1}{3} (1+\sqrt{x})^{3/2} - 1$$