

Lesson 4: Integration By Substitution I

Last time, when $\int h(x) dx = \int f(u(x)) \cdot u'(x) dx$

(1) Choose $u(x)$

(2) Then differentiate to get $du = u'(x) dx$

To produce a new integral

$$\int h(x) dx = \int f(u) du$$

Then integrate and plug back $u(x)$ to get our answer.

Example 1: Compute the following integrals:

(a) $\int 5 \sin(3t) \cos^8(3t) dt = \int 5 \sin(3t) \cdot (\cos(3t))^8 dt$

$$\begin{aligned} u &= \cos(3t) \\ du &= -\sin(3t) \cdot 3 dt \\ \frac{du}{-3\sin(3t)} &= dt \end{aligned} \quad \int 5 \cancel{\sin(3t)} \cdot u^8 \cdot \frac{du}{-3\cancel{\sin(3t)}}$$

$$= -\frac{5}{3} \int u^8 du = -\frac{5}{3} \cdot \frac{u^9}{9} + C$$

$$= -\frac{5}{27} \cos^9(3t) + C$$

(b) $\int 8e^{7x} \csc^2(e^{7x}) dx$

$$\begin{aligned} u &= e^{7x} \\ du &= 7e^{7x} dx \\ \frac{du}{7e^{7x}} &= dx \end{aligned} \quad \int 8 \cancel{e^{7x}} \csc^2(u) \cdot \frac{du}{7\cancel{e^{7x}}}$$

$$= \frac{8}{7} \int \csc^2(u) du = -\frac{8}{7} \cot(u) + C$$

$$= -\frac{8}{7} \cot(e^{7x}) + C$$

Sometimes we need to do extra work with u , as shown in the next example(s).

Example 2: Compute the following integrals:

$$\textcircled{a} \int x \sqrt{x-5} dx \quad \frac{u=x-5}{du=dx} \int x \cdot \sqrt{u} du$$

Issue: Our integral has x and u !
Can I rewrite x into some function of u ? Yes
 $u=x-5 \Leftrightarrow u+5=x$

$$\begin{aligned} &= \int (u+5)u^{1/2} du = \int (u^{3/2} + 5u^{1/2}) du \\ &= \frac{2}{5} u^{5/2} + 5 \cdot \frac{2}{3} u^{3/2} + C \\ &= \frac{2}{5} (x-5)^{5/2} + \frac{10}{3} (x-5)^{3/2} + C \end{aligned}$$

$$\begin{aligned} \textcircled{b} \int \frac{2x}{(x+7)^{3/2}} dx & \quad \frac{u=x+7 \Leftrightarrow x=u-7}{du=dx} \int \frac{2(u-7)}{u^{3/2}} du \\ &= \int (2u^{-1/2} - 14u^{-3/2}) du \\ &= 2 \cdot \frac{2}{1} u^{1/2} - 14 \cdot \frac{2}{-1} u^{-1/2} + C \\ &= 4(x+7)^{1/2} + 28(x+7)^{-1/2} + C \end{aligned}$$

Now let's do integration by substitution for definite integrals.

Example 3: Compute

$$\int_2^4 x \sin(x^2) dx \quad \frac{u=x^2}{du=2x dx} \int x \sin(u) \frac{du}{2x} = \int \frac{1}{2} \sin(u) du$$

$\frac{du}{2x} = dx$

Issue: What do I do with the limit values? i.e. S_2^4

$$\text{Well } \int_2^4 x \sin(x^2) dx = \int_{x=2}^{x=4} x \sin(x^2) dx$$

Method 1: Changing $x=2$ and $x=4$ using $u=x^2$

$$\int_{x=2}^{x=4} x \sin(x^2) dx \quad \begin{array}{c} \downarrow \\ u=4 \end{array} \quad \begin{array}{c} \downarrow \\ u=16 \end{array} \quad \frac{u=x^2}{\frac{du}{2x}=dx} \quad \int_{u=4}^{u=16} \sin(u) \frac{du}{2}$$

$$= \left[-\frac{\cos(u)}{2} \right]_{u=4}^{u=16} = -\frac{1}{2} (\cos(16) - \cos(4))$$

Method 2: To treat the integral after the u -sub as indefinite and when you plug u back evaluate with original bounds.

$$\int_{x=2}^{x=4} x \sin(x^2) dx = \frac{u=x^2}{\frac{du}{2x}=dx} \int \frac{\sin(u)}{2} du$$

$$= \left[-\frac{\cos(u)}{2} = -\frac{\cos(x^2)}{2} \right]_{x=2}^{x=4}$$

$$= -\frac{1}{2} (\cos(16) - \cos(4))$$

Example 4: If the area of the region under the curve $y = \frac{1}{\sqrt{9x+4}}$ over the interval $0 \leq x \leq a$ is 6, then what is a ?

i.e. Solve $\int_0^a \frac{1}{\sqrt{9x+4}} dx = 6$ for a ,

$$6 = \int_0^a \frac{1}{\sqrt{9x+4}} dx \quad \frac{u=9x+4}{\frac{du}{9}=dx} \quad \int \frac{1}{\sqrt{u}} \cdot \frac{du}{9} = \frac{1}{9} \int u^{-1/2} du$$

$$= \frac{1}{9} \cdot \frac{2}{1} u^{1/2} = \frac{2}{9} (9x+4)^{1/2} \Big|_0^a = \frac{2}{9} (9a+4)^{1/2} - \frac{2}{9} (4)^{1/2}$$

$$= \frac{2}{9} (9a+4)^{1/2} - \frac{4}{9}$$

$$\frac{58}{9} = \frac{2}{9} (9a+4)^{1/2}$$

$$\frac{9}{2} \cdot \frac{58}{9} = \frac{9}{2} \cdot \frac{2}{9} (9a+4)^{1/2}$$

$$29 = (9a+4)^{1/2}$$

$$(29)^2 = 9a+4$$

$$841 = 9a+4$$

$$837 = 9a$$

$$a = \frac{837}{9} = 93$$