

Lesson 7: Integration by Parts I

Recall the Product Rule,

$$(uv)' = u'v + uv'$$

what if we integrate both sides with respect to x .

$$\int (uv)' dx = \int (u'v + uv') dx$$

$$\int (uv)' dx = \int u'v dx + \int uv' dx$$

$$uv = \int u'v dx + \int uv' dx$$

Remember $u' = \frac{du}{dx}$ and $v' = \frac{dv}{dx}$. So

$$uv = \int v \frac{du}{dx} dx + \int u \frac{dv}{dx} dx$$

$$uv = \int v du + \int u dv$$

$$uv - \int v du = \int u dv$$

Interesting enough is that for many cases an integral will have the form

$$\int u dv$$

This technique of turning one integral into another is called **Integration by Parts**.

It's formula is: $\int u dv = uv - \int v du$

To use this technique

- Choose u to be the one to differentiate
- Choose dv to be integrated

Remark: Sometimes picking u and dv can be tricky. There is an acronym that makes picking u easier. Think of it as a sort of order of operation for choosing u .

L - Logarithmic

A - Algebraic

T - Trigonometric

E - Exponential

Example 1: Evaluate

$$\begin{aligned} \textcircled{a} \int x \ln x \, dx & \quad \frac{u = \ln x}{du = \frac{1}{x} dx} \quad \frac{dv = x \, dx}{v = \frac{x^2}{2}} \quad uv - \int v \, du \\ & = \frac{x^2}{2} \ln x - \int \frac{x^2}{2} \cdot \frac{1}{x} \, dx = \frac{x^2}{2} \ln x - \int \frac{x}{2} \, dx \\ & = \frac{x^2}{2} \ln x - \frac{1}{2} \cdot \frac{x^2}{2} + C = \frac{x^2}{2} \ln x - \frac{x^2}{4} + C \end{aligned}$$

$$\begin{aligned} \textcircled{b} \int x \cos x \, dx & \quad \frac{u = x}{du = dx} \quad \frac{dv = \cos x \, dx}{v = \sin x} \quad uv - \int v \, du \\ & = x \sin x - \int \sin x \, dx = x \sin x - (-\cos x) + C \\ & = x \sin x + \cos x + C \end{aligned}$$

$$\begin{aligned} \textcircled{c} \int x e^{2x} \, dx & \quad \frac{u = x}{du = dx} \quad \frac{dv = e^{2x} \, dx}{v = \frac{1}{2} e^{2x}} \quad uv - \int v \, du \\ & = \frac{x}{2} e^{2x} - \int \frac{1}{2} e^{2x} \, dx = \frac{x}{2} e^{2x} - \frac{1}{2} \cdot \frac{1}{2} e^{2x} + C \\ & \quad \text{found by } u\text{-sub} \\ & = \frac{x}{2} e^{2x} - \frac{1}{4} e^{2x} + C \end{aligned}$$

Example 2: Evaluate

$$\begin{aligned} \textcircled{a} \int_0^{\pi/2} 8x \sin x \, dx & \quad \begin{array}{l} u=8x \quad dv=\sin x \, dx \\ du=8 \, dx \quad v=-\cos x \end{array} \\ & = -8x \cos x \Big|_0^{\pi/2} - \int_0^{\pi/2} -8 \cos x \, dx \\ & = -8x \cos x \Big|_0^{\pi/2} + \int_0^{\pi/2} 8 \cos x \, dx \\ & = -8x \cos x \Big|_0^{\pi/2} + 8 \sin x \Big|_0^{\pi/2} \\ & = -\frac{8\pi}{2} \cos\left(\frac{\pi}{2}\right) - (-8(0) \cos(0)) + 8 \sin\left(\frac{\pi}{2}\right) - 8 \sin(0) \\ & = 0 - 0 + 8 - 0 = 8 \end{aligned}$$

$$\begin{aligned} \textcircled{b} \int_1^e x \ln^3 \sqrt{x} \, dx & = \int_1^e x \ln(x^{1/2}) \, dx = \int_1^e \frac{1}{3} x \ln x \, dx \\ & \quad \begin{array}{l} u=\ln x \quad dv=\frac{1}{3} x \, dx \\ du=\frac{1}{x} \, dx \quad v=\frac{1}{3} \cdot \frac{x^2}{2} \end{array} \\ & = \frac{x^2}{6} \ln x \Big|_1^e - \int_1^e \frac{x}{6} \, dx = \frac{x^2}{6} \ln x \Big|_1^e - \frac{x^2}{12} \Big|_1^e \\ & = \frac{e^2}{6} \ln e - \frac{1}{6} \ln(1) - \left(\frac{e^2}{12} - \frac{1}{12} \right) \\ & = \frac{e^2}{6} - 0 - \frac{e^2}{12} + \frac{1}{12} \end{aligned}$$

$$\begin{aligned} \textcircled{c} \int_1^e \frac{\ln x}{x^4} \, dx & = \int_1^e x^{-4} \ln x \, dx \quad \begin{array}{l} u=\ln x \quad dv=x^{-4} \, dx \\ du=\frac{1}{x} \, dx \quad v=\frac{x^{-3}}{-3} = -\frac{1}{3x^3} \end{array} \\ & = -\frac{1}{3x^3} \ln x \Big|_1^e - \int_1^e -\frac{1}{3x^3} \cdot \frac{1}{x} \, dx = -\frac{\ln x}{3x^3} \Big|_1^e + \int_1^e \frac{1}{3x^4} \, dx \\ & = -\frac{\ln x}{3x^3} \Big|_1^e + \frac{1}{3} \int_1^e x^{-4} \, dx = -\frac{\ln x}{3x^3} \Big|_1^e + \frac{1}{3} \frac{x^{-3}}{-3} \Big|_1^e \\ & = -\frac{\ln e}{3e^3} - \left(-\frac{\ln(1)}{3 \cdot 1} \right) - \frac{1}{9} \left((e)^{-3} - 1^{-3} \right) \\ & = -\frac{1}{3e^3} - 0 - \frac{1}{9} \left(\frac{1}{e^3} - 1 \right) = -\frac{1}{3e^3} - \frac{1}{9e^3} + \frac{1}{9} \end{aligned}$$

And sometimes you have to apply this method multiple times.

Example 3: Evaluate

$$\int 12x (\ln(4x))^2 dx \quad \begin{array}{l} u = (\ln(4x))^2 \\ du = 2(\ln(4x)) \cdot \frac{1}{4x} \cdot 4 dx \\ du = 2(\ln(4x)) dx \end{array} \quad \begin{array}{l} dv = 12x dx \\ v = \frac{12x^2}{2} \\ v = 6x^2 \end{array}$$

$$= 6x^2 (\ln(4x))^2 - \int 6x^2 \cdot \frac{2(\ln(4x))}{x} dx$$

$$= 6x^2 (\ln(4x))^2 - \int 12x \ln(4x) dx$$

Note to integrate this we need to do another integration by parts

$$\int 12x \ln(4x) dx \quad \begin{array}{l} u = \ln(4x) \\ du = \frac{1}{4x} \cdot 4 dx \\ du = \frac{1}{x} dx \end{array} \quad \begin{array}{l} dv = 12x dx \\ v = \frac{12x^2}{2} \\ v = 6x^2 \end{array}$$

$$= 6x^2 \ln(4x) - \int 6x^2 \cdot \frac{1}{x} dx$$

$$= 6x^2 \ln(4x) - \int 6x dx$$

$$= 6x^2 \ln(4x) - \frac{6x^2}{2} = 6x^2 \ln(4x) - 3x^2$$

$$\downarrow$$

$$= 6x^2 (\ln(4x))^2 - (6x^2 \ln(4x) - 3x^2) + C$$

$$= 6x^2 (\ln(4x))^2 - 6x^2 \ln(4x) + 6x^2 + C$$