Lesson 10: First-Order Linear Differential Equations

From the Problem Set (posted online),

Example 1: Find the general solution of the given differential equation

$$(x-7)y' + y = x^2 - 10$$

Follow the Steps outlined in Lesson 9.

① $$\frac{(x-7)y'}{x-7} + \frac{1}{x-7}y = \frac{x^2-10}{x-7}$$

$$y' + \frac{1}{x-7}y = \frac{x^2-10}{x-7}$$

② $P(x) = \dfrac{1}{x-7}$ $\quad Q(x) = \dfrac{x^2-10}{x-7}$

③ $$u(x) = \exp[\textstyle\int P(x)\,dx] = \exp\left[\int \frac{1}{x-7}\,dx\right] = \exp[\ln(x-7)]$$

$$= x-7$$

④ $$y \cdot u(x) = \int Q(x)\,u(x)\,dx$$

$$y(x-7) = \int \frac{(x^2-10)\cdot(x-7)}{(x-7)}\,dx$$

⑤ $$y(x-7) = \int (x^2-10)\,dx$$

$$y(x-7) = \frac{x^3}{3} - 10x + C$$

$$y(x-7) = \frac{x^3 - 30x + C}{3}$$

⑥ $y = \dfrac{x^3 - 30x + C}{3(x-7)}$

Example 2: Jill owns an electronics store with storage capacity for 70 iPads. She currently has 55 iPads in inventory and determines that they are selling at a daily rate equal to 14% of the avaliable capacity. When will Jill sell out of iPads?

Set-up: $\dfrac{dN}{dt} = -\begin{bmatrix}\text{Daily rate} \\ \text{of Sales}\end{bmatrix} = -0.14(70-N) = -9.8 + 0.14N$

① $\dfrac{dN}{dt} - 0.14N = -9.8$

② $P(t) = -0.14 \qquad Q(t) = -9.8$

③ $u(t) = \exp[\int P(t)dt] = \exp[\int -0.14\,dt] = e^{-0.14t}$

④ $y \cdot u(t) = \int Q(t)u(t)\,dt$

$y \exp[-0.14t] = \int -9.8 \exp[-0.14t]\,dt$

⑤ Use a u-substition on RHS.

$u = -0.14t \qquad du = -0.14\,dt$

$y\exp[-0.14t] = \int \dfrac{-9.8}{-0.14} e^u\,du$

$y\exp[-0.14t] = 70e^u + C$

$y\exp[-0.14t] = 70\exp[-0.14t] + C$

⑥ $y = 70 + C\exp[0.14t]$

When $t=0$, $N=55$

$$55 = 70 + C$$
$$-15 = C \qquad \Rightarrow N = 70 - 15\exp[0.14t]$$

Solve $N(t)=0$ for t.

$$0 = 70 - 15e^{0.14t}$$

$$70 = 15e^{0.14t}$$

$$\frac{14}{3} = \frac{70}{15} = e^{0.14t}$$

$$\ln\left(\frac{14}{3}\right) = 0.14t$$

$$t = \frac{1}{0.14}\ln\left(\frac{14}{3}\right) \approx 11.003$$

Example 3: A corporation is initially worth 5 million dollars and is growing in value, V, by 26% each year, and is additionally gaining 20% of a growing market estimated at

$$100e^{0.26t} \text{ million dollars}$$

where t is the number of years the company has existed. Approximated the value of the company after 6 years.

Set-Up: $V' = 0.26V + 0.2(100e^{0.26t})$
$$= 0.26V + 20e^{0.26t}$$

① $V' - 0.26V = 20e^{0.26t}$

② $P(t) = -0.26 \qquad Q(t) = 20e^{0.26t}$

③ $u(t) = \exp[\int P(t)dt] = \exp[\int -0.26\,dt] = e^{-0.26t}$

④ $y \cdot u(t) = \int Q(t)u(t)\,dt$

$$ye^{-0.26t} = \int 20e^{0.26t}e^{-0.26t}\,dt$$

⑤ $ye^{-0.26t} = \int 20\,dt$

$ye^{-0.26t} = 20t + C$

⑥ $y = (20t + C)e^{0.26t}$

When $V(0) = 5$,

$5 = V(0) = C \Rightarrow V = (20t + 5)e^{0.26t}$

$V(6) = (120 + 5)e^{0.26(6)} \approx \$595$