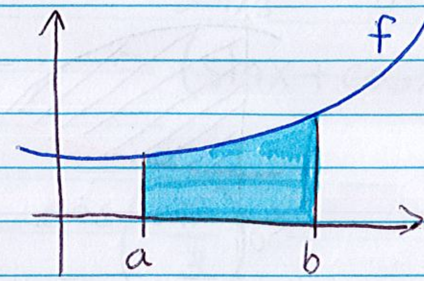
Lesson 11: Area Between Two Curves

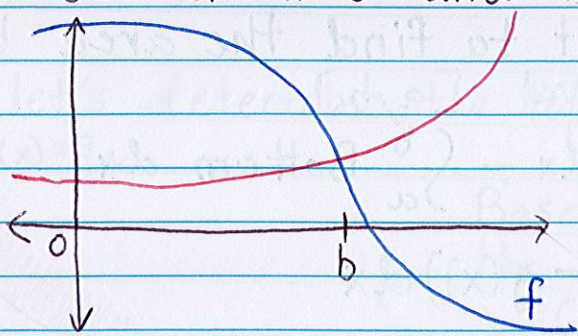
Recall from Calculus I that the definite integral has a geometric meaning, namely the area under a curve.

i.e. $\int_a^b f(x)\,dx$

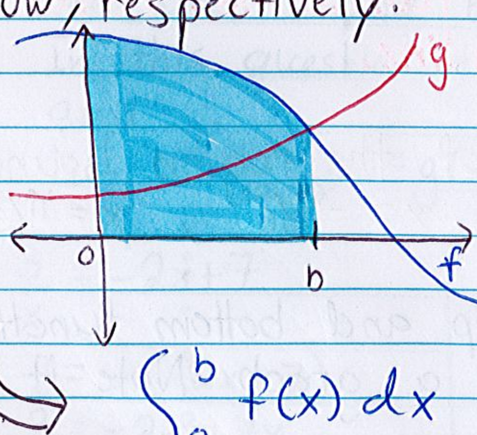


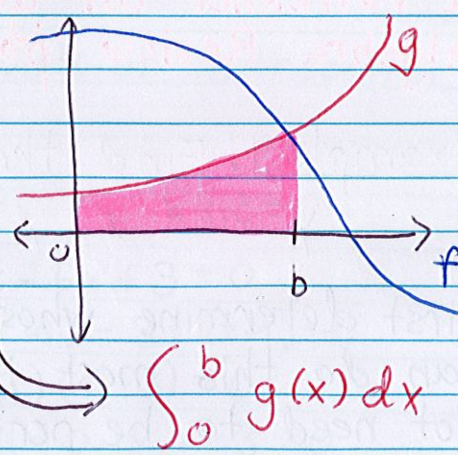
In this Lesson, we want the area BETWEEN 2 curves.

Consider the graphs of f and g, as shown below, and say we want to calculate the area bounded by the two curves between $x=0$ and $x=b$.



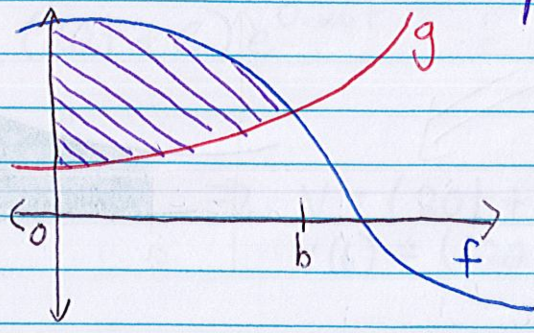
If we calculate the area under each curve separately we find the blue and red areas in the two graphs below, respectively.





$\int_0^b f(x)\,dx$ $\qquad$ $\int_0^b g(x)\,dx$

Looking at these graphs we can see the graph on the right is what we don't want. So if we subtract the red area from the blue area, we get the area between the two curves. i.e. purple area.



Using integrals, we have

$$\text{Area} = \int_0^b f(x)dx - \int_0^b g(x)dx = \int_0^b (f(x) - g(x))\,dx$$

Generally if we want to find the area between two curves on an interval $[a,b]$

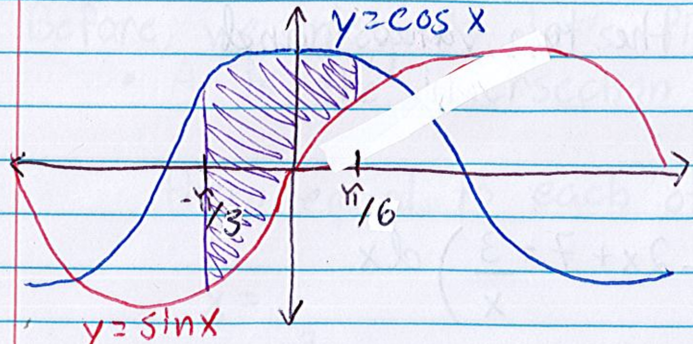
$$\text{Area} = \int_a^b \text{Top}\,dx - \int_a^b \text{Bottom}\,dx$$

$$= \int_a^b (f(x) - g(x))\,dx$$

With all of these problems, you want to draw the graph corresponding with the problem. If you need a refresher on graphing functions, refer to the Algebra Review posted on line.

Example 1: Find the area of the region bounded by
$y = \sin x$, $y = \cos x$, $x = -\pi/3$, $x = \pi/6$

First determine whose the top and bottom function. You can do this most times via a graph. Note it does not need to be perfect.



Based on the graph, we have the integral,

$$\int_{-\pi/3}^{\pi/6} (\cos x - \sin x)\,dx$$

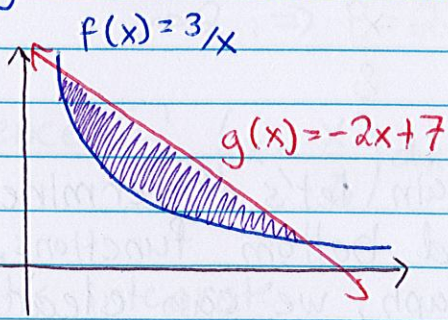
$$= (\sin x + \cos x)\Big|_{-\pi/3}^{\pi/6}$$

$$= \sin\left(\frac{\pi}{6}\right) + \cos\left(\frac{\pi}{6}\right) - \sin\left(-\frac{\pi}{3}\right) - \cos\left(-\frac{\pi}{3}\right)$$

$$= \frac{1}{2} + \frac{\sqrt{3}}{2} - \left(-\frac{\sqrt{3}}{2}\right) - \frac{1}{2} = \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} = \sqrt{3}$$

Example 2: Find the area of the region bounded by $f(x) = \frac{3}{x}$ and $g(x) = -2x + 7$

Again let's determine the top and bottom functions.



Based on the graph, we have the integral,

$$\int_a^b \left(-2x + 7 - \frac{3}{x}\right)dx$$

But what are a and b?

In this question it's the points of intersection of f and g.

i.e. when $f(x) = g(x)$

$$\frac{3}{x} = -2x + 7$$
$$3 = x(-2x + 7)$$
$$3 = -2x^2 + 7x$$
$$2x^2 - 7x + 3 = 0$$
$$2x^2 - 7x + 3 = 0$$

$$2x^2 - x - 6x + 3 = 0$$
$$x(2x - 1) - 3(2x - 1) = 0$$
$$(2x - 1)(x - 3) = 0$$

$2x - 1 = 0$, $x = \frac{1}{2}$ | $x - 3 = 0$, $x = 3$

So a is the smaller of the two values and b is the largest.

Hence $a = 1/2$, $b = 3$.

Which gives us $\int_{1/2}^{3} \left(-2x + 7 - \frac{3}{x}\right) dx$

Now integrate.

$$\int_{1/2}^{3} \left(-2x+7-\frac{3}{x}\right)dx = \left(\frac{-2x^2}{2} + 7x - 3\ln|x|\right)\Big]_{1/2}^{3}$$

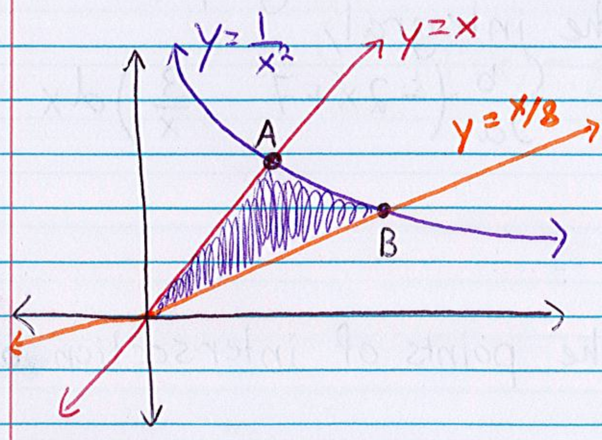
$$= \left(-x^2 + 7x - 3\ln|x|\right)\Big]_{1/2}^{3}$$

$$= -3^2 + 7(3) - 3\ln(3) - \left(-\left(\frac{1}{2}\right)^2 + 7\left(\frac{1}{2}\right) - 3\ln\left(\frac{1}{2}\right)\right)$$

$$= \frac{35}{4} + 3\ln\left(\frac{1}{2}\right) - 3\ln 3$$

Example 3: Find the area of the region R bounded by

$$y = \frac{1}{x^2}, \quad y = x, \quad y = \frac{x}{8}$$



Again let's determine the top and bottom functions. In the graph, we can clearly see that

Bottom $\Rightarrow$ $y = \frac{x}{8}$

But who is the Top? It depends.

From O to A, $y = x$ is top and
From A to B, $y = \frac{1}{x^2}$ is top.
Which yields 2 integrals.

So $\int_{0}^{A} \left(x - \frac{x}{8}\right) dx + \int_{A}^{B} \left(\frac{1}{x^2} - \frac{x}{8}\right) dx$

Before, we integrate let's find A and B.

- A is the intersection of $y=x$ and $y=\frac{1}{x^2}$. So set them equal to each other, to find A.

$$x=\frac{1}{x^2}$$
$$x^3=1$$
$$x=1 \Rightarrow A=1$$

- B is the intersection of $y=\frac{x}{8}$ and $y=\frac{1}{x^2}$. So set them equal to each other, to find B.

$$\frac{x}{8}=\frac{1}{x^2}$$
$$x^3=8$$
$$x=2 \Rightarrow B=2$$

Hence $\int_0^1 \left(x-\frac{x}{8}\right)dx + \int_1^2 \left(\frac{1}{x^2}-\frac{x}{8}\right)dx$

Let's integrate

$$\int_0^1 \left(\frac{7}{8}x\right)dx + \int_1^2 \left(x^{-2}-\frac{1}{8}x\right)dx$$
$$= \frac{7}{8}\cdot\frac{x^2}{2}\Big]_0^1 + \left(\frac{x^{-1}}{-1}-\frac{1}{8}\cdot\frac{x^2}{2}\right)\Big]_1^2$$
$$= \frac{7}{16}x^2\Big]_0^1 + \left(-\frac{1}{x}-\frac{1}{16}x^2\right)\Big]_1^2$$
$$= \frac{7}{16}(1)^2-\frac{7}{16}(0)^2+\left(\frac{-1}{2}-\frac{1}{16}(2)^2\right)-\left(\frac{-1}{1}-\frac{1}{16}(1)^2\right)$$
$$= \frac{3}{4}$$

Example 3 (continued): Find the vertical line that divides the area, found previously, in half.

i.e. We want $\int_0^K \text{Region}\, dx = \frac{1}{2}\left(\frac{3}{4}\right) = \frac{3}{8}$ and solve for K.

The question is whether k lies between (0,1) or (1,2).
Previously, we found

$$\int_0^1 \left(\frac{7}{8}x\right) dx = \frac{7}{16} \quad \text{and} \quad \int_1^2 \left(\frac{1}{x^2} - \frac{x}{8}\right) dx = \frac{5}{16}$$

So is $3/8 < 7/16$? Yes.

That means the half-way pt, k, is in the interval (0,1)

$$\int_0^k \left(\frac{7}{8}x\right) dx = \frac{3}{8}$$

$$\frac{7}{16}x^2\Big]_0^k = \frac{3}{8}$$

$$\frac{7}{16}k^2 - \frac{7}{16}(0)^2 = \frac{3}{8}$$

$$\frac{7}{16}k^2 = \frac{3}{8}$$

$$k^2 = \frac{3}{8} \cdot \frac{16}{7} = \frac{6}{7}$$

$$k = \pm\sqrt{\frac{6}{7}}$$

Since we are in the interval (0,1), k can only be

$$k = \sqrt{\frac{6}{7}}$$