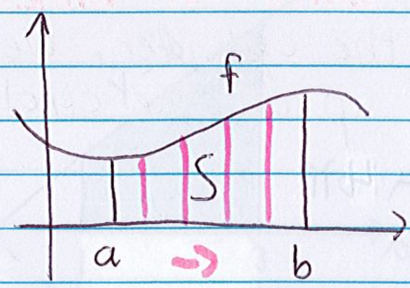
Lesson 12: Volume of Solids of Revolution

In geometry, we first talked about the concept of area. We did this by going over all the formulas for the area of different polygons.

In Calculus I, we learned about integration as a new technique for calculating area under a curve.

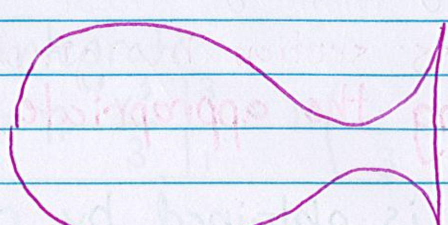
i.e. $\int_a^b f(x)dx = F(b) - F(a)$

We learned how to take some region between a curve and an axis, or between 2 curves, and find its area by integration. Essentially, finding a length and sending it across the region.



To see the animated version of this image refer to GeoGebra file "MA 16020 Lesson 12 Calculus 1 Recap"

But in geometry we also learned about 3-D figures, like cubes and prisms. We described the volume of these objects, or the amount of 3-D space that they contain, and we had useful formulas to calculate this as well. But once curves get involved all of these formulas are useless.



There is no formula for this.

Luckily, just the way we run a line segment across a 2-D region to calculate its area, we can run a plane region, or cross section, across a 3-D region to calculate

its volume.
i.e. running a 2-D plane across a 3D volume.

This is also integration, just with an extra dimension. Instead of adding up tiny rectangles under a curve in the limit of infinitely thin rectangles, we are adding up infinitely thin cross sections, which we can call disks (Lesson 12) or washers (Lesson 13).

Since each of these disks is a 2-D area, taking the integral of an area function will gives us volume.

For example, let's look at a cyclinder
Open GeoGebra file titled
"MA 16020 Lesson 12 Intro for Disks"
You can see the red circle in the cylinder. We can think of our integral to be summing up all the circles.

$$\text{Volume of Cylinder} = \underbrace{\int_{-2}^{2}}_{\text{height}} \underbrace{2^2\pi}_{\text{area of a circle}}\, dx = 16\pi$$

which checks out because our geometry formula states
$V = \pi r^2 h$ where $r=2$, $h=4$
$= 16\pi$

So in the case of a cyllinder, this might be overkill. But this is the way we want to think of these questions.
Essentially find the cross section by graphing the lines given and applying the appropriate formula:

$\boxed{\text{Disk Method}}$ If the volume of the solid is obtained by rotating $f(x)$ about the x-axis on the interval $a \le x \le b$ is given by

$$V = \pi \int_a^b [f(x)]^2\, dx$$

Disk Method If the volume of the solid is obtained by rotating $g(y)$ about the y-axis on the interval $c \le y \le d$ in a similar way:

$$V = \pi \int_c^d [g(y)]^2 \, dy$$

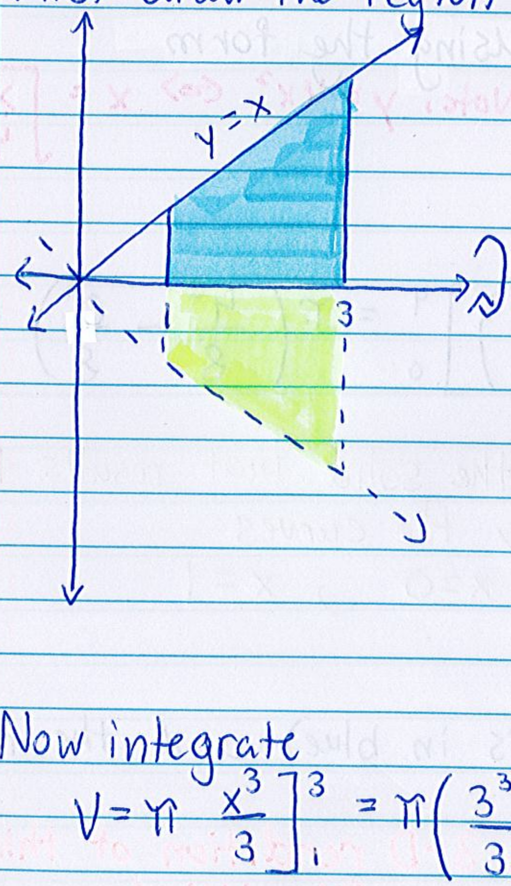
Note this π in both formulas comes from the fact we are playing with Disks.

Example 1: Find the volume of the solid that results by revolving the region enclosed by the curves

$y = x$; $y = 0$; $x = 1$; $x = 3$

about x-axis.

First draw the region (which is in blue) and the reflection (shown in yellow).



To see the 3-D rendition of this graph open GeoGebra file titled "MA 16020: Lesson 12 Example 1 (xAxis)"

Using the formulas, we have the integral

$$V = \pi \int_1^3 (x)^2 \, dx$$

Now integrate

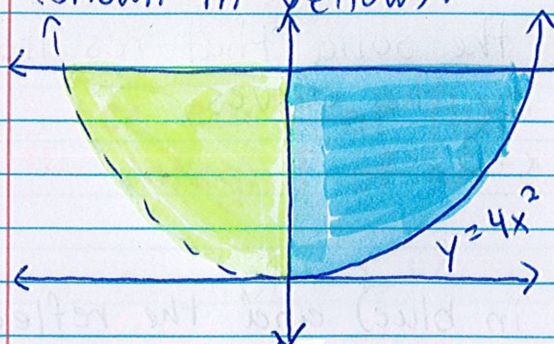
$$V = \pi \left. \frac{x^3}{3} \right]_1^3 = \pi \left(\frac{3^3}{3} - \frac{1^3}{3} \right) = \frac{26}{3} \pi$$

Example 2: Find the volume of the solid that results by revolving the region in the first quadrant enclosed by the curves

$$y = 4x^2 \; ; \; x = 0 \; ; \; y = 4$$

about y-axis.

First draw the region (which is in blue) and the reflection (shown in yellow).



To see the 3-D rendition of this graph open GeoGebra file titled "MA 16020: Lesson 12 Example 2 (yAxis)"

Using the formulas, we have the integral

$$V = \pi \int_0^4 \left(\sqrt{\frac{y}{4}}\right)^2 dy$$

Note: $y = 4x^2 \iff x = \sqrt{\frac{y}{4}}$

Now integrate

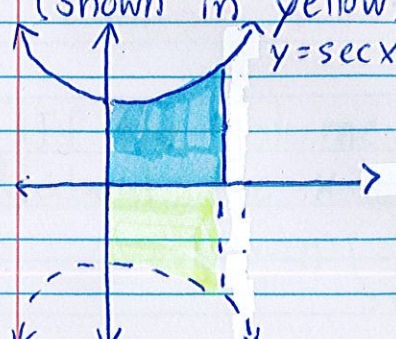
$$V = \pi \int_0^4 \frac{y}{4}\, dy = \pi\left(\frac{1}{4}\frac{y^2}{2}\right)\Big]_0^4 = \pi\left(\frac{4^2}{8} - \frac{0^2}{8}\right) = 2\pi$$

Example 3: Find the volume of the solid that results by revolving the region enclosed by the curves

$$y = \sec x \; ; \; y = 0 \; ; \; x = 0 \; ; \; x = 1$$

about the x-axis

First draw the region (which is in blue) and the reflection (shown in yellow)



To see the 3-D rendition of this graph open GeoGebra file titled "MA 16020: Lesson 12 Example 3 (xAxis)"

Using the formulas, we have the integral

$$V = \pi \int_0^1 (\sec(x))^2 \, dx$$

Now integrate.

$$V = \pi \int_0^1 \sec^2 x \, dx = \pi \tan x \Big]_0^1 = \pi(\tan(1) - \tan(0))$$

$$= \pi \cdot \tan(1)$$