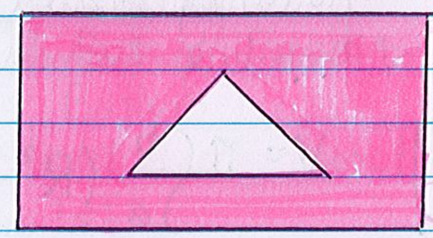
Lesson 13: Volume By Revolution

Last Time, we talked about
- How Geometry gave us formulas for simple shapes and solids to find their area or volume, and
- How Integration can allow us to find area or volume of ANYTHING!

How? We introduced this notion of cross-sections which can be of the form of
- Disks (Lesson 12)
- Washers (Lesson 13)

Let's do a bit more Geometry. In particular, let's calculate the area of a shaded region.

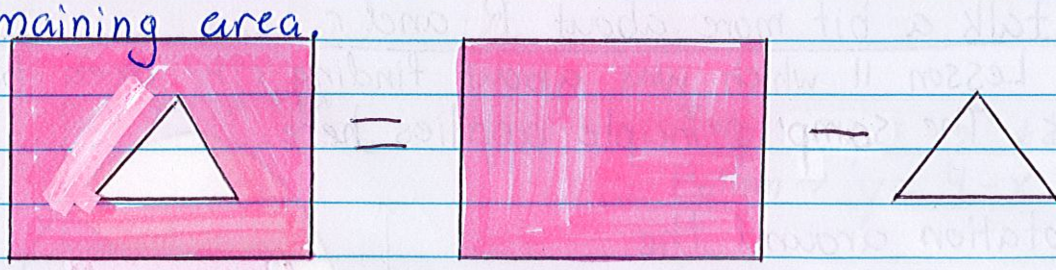
Suppose we are asked to find the area of rectangle with a triangle missing from the middle. (i.e. the pink area)



How do we calculate that area?

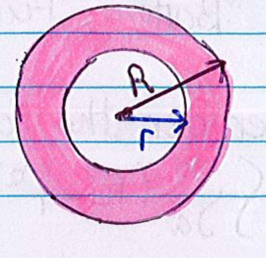
First, we would find the area of the rectangle and the area of the triangle separately.

Then we would subtract these two values to find the remaining area.



What if we did this with disks?

Let's find the area of the pink annulus.



The area of the outer circle is πR^2, and the area of the inner circle is πr^2.

So if we subtract the two, we get
$$\pi R^2 - \pi r^2 = \pi(R^2 - r^2)$$

In this lesson, we are going to play with disks, but remove a portion of it. This method is called the washer method.

Washer Method Formula

Since we are just cutting out the middle of the solid, we choose dx or dy in the same way as the disk method.

- Rotating around x-axis ⇒ "dx" problem
- Rotating around y-axis ⇒ "dy" problem

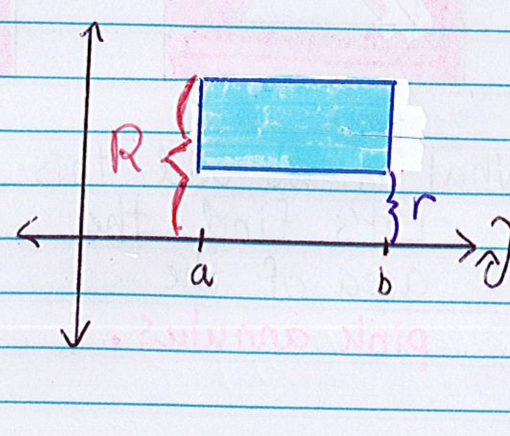
Washer Method
$$V = \pi \int_a^b (R^2 - r^2)\,dx$$
where a and b are bounds of the region we are rotating.

Note: • R is the farthest from the axis of rotation, and
• r is the closest

Let's talk a bit more about R and r

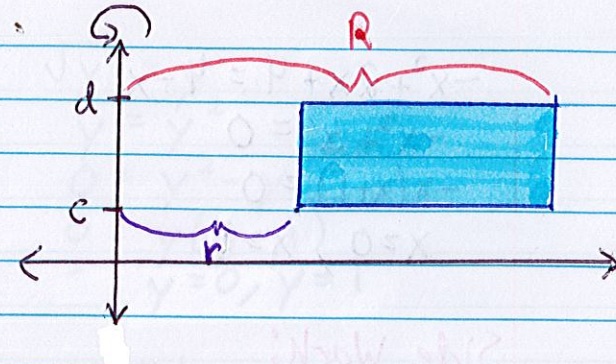
Recall Lesson 11 which was about finding the area between 2 curves. The same principle applies here.

For rotation around the x-axis,
- R is "Top" Function
- r is "Bottom" Function



Just remember the formula is
$$V = \pi \int_a^b (R^2 - r^2)\,dx$$

For rotation around the y-axis,
- R is the "Right" Function
- r is the "Left" Function



Just remember the formula is

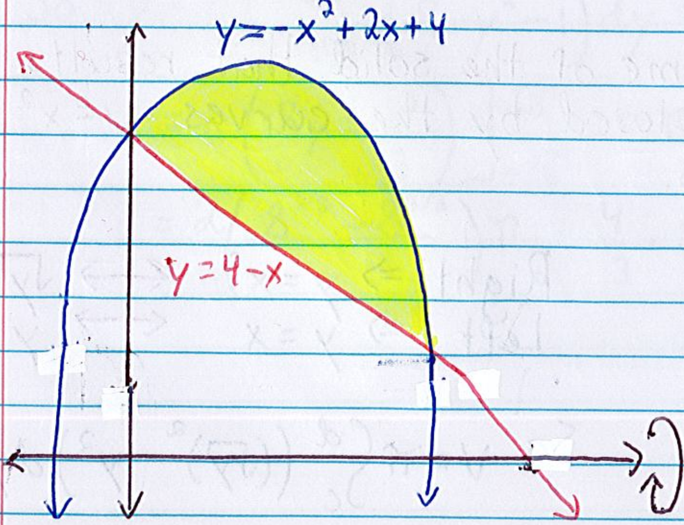
$$V = \pi \int_c^d (R^2 - r^2)\, dy$$

If you need a refresher of "Right Minus Left", read the pdf titled "Lesson 11: Bonus Material"

Again, we proceed with Washer Problems by
1. Draw the region
2. Determine which axis you are rotating on
   - (a) If x-axis: Determine Top and Bottom Function
     - (i) R is Top
     - (ii) r is Bottom
   - (b) If y-axis: Determine Right and Left Function
     - (i) R is Right
     - (ii) r is Left
3. Apply the Washer Formula

$$V = \pi \int_a^b (R^2 - r^2)\, dx$$

<u>Example 1</u>: Find the volume of the solid that results by revolving the region enclosed by the curves $y = -x^2 + 2x + 4$ and $y = 4 - x$ about the x-axis.



Top $\Rightarrow$ $y = -x^2 + 2x + 4$
Bottom $\Rightarrow$ $y = 4 - x$

So

$$V = \pi \int_a^b \left((-x^2 + 2x + 4)^2 - (4 - x)^2 \right) dx$$

To find a and b set $y = 4 - x$ and $y = -x^2 + 2x + 4$ equal.

$$-x^2+2x+4 = 4-x$$
$$-x^2+3x = 0$$
$$-x(x-3) = 0$$
$$x=0,\ x=3$$

So $V = \pi\int_0^1 \left((-x^2+2x+4)^2-(4-x)^2\right)dx$

Side Work:

	$-x^2$	$2x$	4
$-x^2$	x^4	$-2x^3$	$-4x^2$
$2x$	$-2x^3$	$4x^2$	$8x$
4	$-4x^2$	$8x$	16

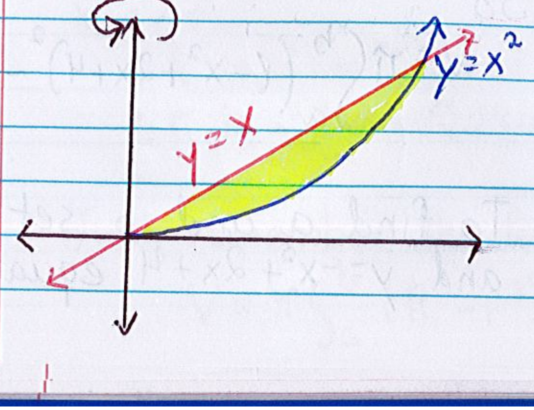
$\Rightarrow (-x^2+2x+4)^2 = x^4-4x^3-4x^2+16x+16$

	4	$-x$
4	16	$-4x$
$-x$	$-4x$	x^2

$\Rightarrow (4-x)^2 = 16-8x-x^2$

So
$$V = \pi\int_0^3 \left(x^4-4x^3-4x^2+16x+16-16+8x+x^2\right)dx$$
$$= \pi\int_0^3 \left(x^4-4x^3-3x^2+24x\right)dx$$
$$= \pi\left(\frac{x^5}{5}-\frac{4x^4}{4}-\frac{3x^3}{3}+\frac{24x^2}{2}\right)\Bigg]_0^3$$
$$= \pi\left(\frac{x^5}{5}-x^4-x^3+12x^2\right)\Bigg]_0^3$$
$$= \pi\left(\frac{3^5}{5}-(3)^4-(3)^3+12(3)^2\right) = \frac{243}{5}\pi$$

Example 2: Find the volume of the solid that results by revolving the region enclosed by the curves $y=x^2$ and $y=x$ about the y-axis.



Right $\Rightarrow y=x^2 \Longleftrightarrow \sqrt{y}=x$

Left $\Rightarrow y=x \Longleftrightarrow y=x$

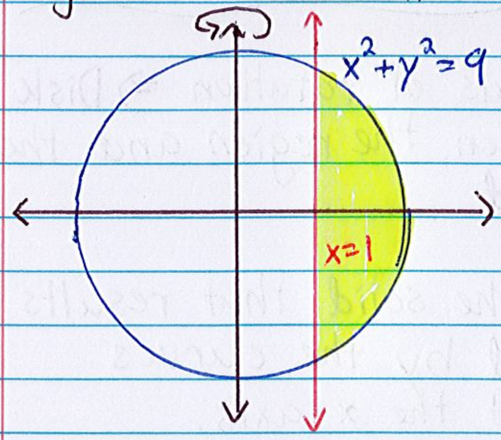
So $V = \pi\int_c^d \left((\sqrt{y})^2-y^2\right)dy$

To find c and d set
$\sqrt{y} = x$ and $y = x$ equal.

$$\sqrt{y} = y$$
$$y = y^2$$
$$0 = y^2 - y$$
$$0 = y(y-1)$$
$$y = 0,\ y = 1$$

So $V = \pi \int_0^1 (y - y^2)\,dy = \pi\left(\frac{y^2}{2} - \frac{y^3}{3}\right)\Big]_0^1 = \pi\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{\pi}{6}$

<u>Example 3</u>: Find the volume of the solid that results by revolving the region inside the circle $x^2 + y^2 = 9$ and to the right of the line $x = 1$ about the y-axis.



Right $\Rightarrow x^2 + y^2 = 9 \longleftrightarrow x = \sqrt{9 - y^2}$
Left $\Rightarrow x = 1$

So $V = \pi \int_c^d \left(\left(\sqrt{9-y^2}\right)^2 - 1^2\right) dy$

To find c and d, plug $x = 1$ into

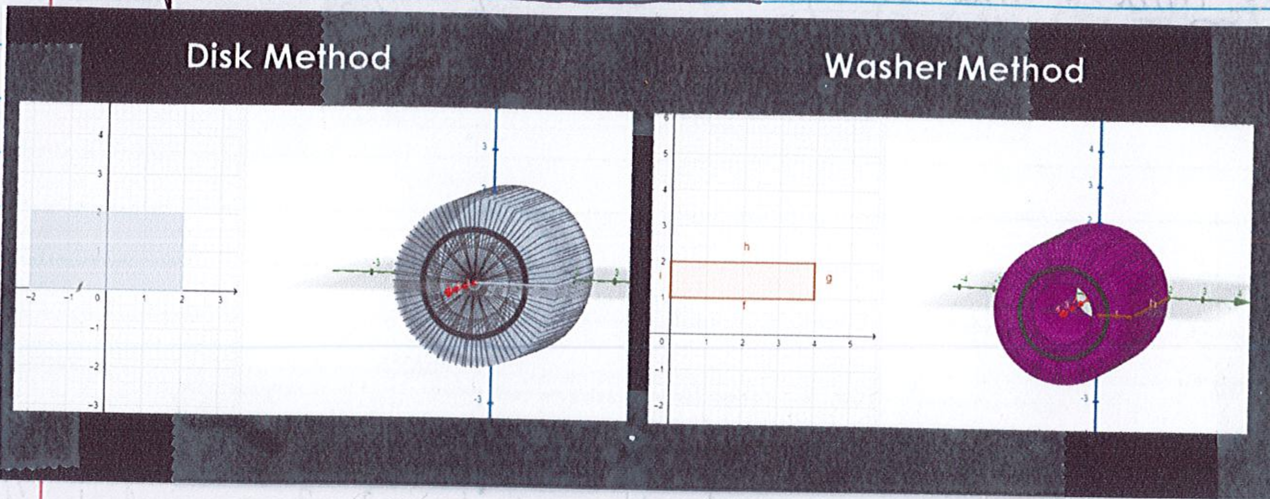
$$x^2 + y^2 = 9$$
$$1 + y^2 = 9$$
$$y^2 = 8$$
$$y = \pm\sqrt{8}$$

So $V = \pi \int_{-\sqrt{8}}^{\sqrt{8}} (9 - y^2 - 1)\,dy = \pi \int_{-\sqrt{8}}^{\sqrt{8}} (8 - y^2)\,dy$

$= \pi\left(8y - \frac{y^3}{3}\right)\Big]_{-\sqrt{8}}^{\sqrt{8}} = \pi\left(8\sqrt{8} - \frac{8^{3/2}}{3} - \left(-8\sqrt{8} + \frac{8^{3/2}}{3}\right)\right)$

$= 2\left(8^{3/2} - \frac{8^{3/2}}{3}\right)\pi = \frac{4}{3}\cdot 8^{3/2}\pi \approx 94.782$

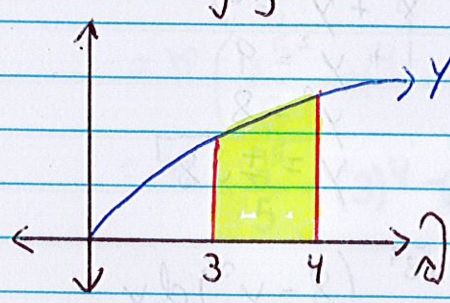
Recap: Disk vs. Washer Method



When do we apply Disk Method or Washer Method?

- When the region "hugs" the axis of rotation ⇒ Disk Method
- When there is a "gap" between the region and the axis of rotation ⇒ Washer Method

Example 4: Find the volume of the solid that results by revolving the region enclosed by the curves $y = \sqrt{5x} + \sqrt{\frac{x}{5}}$, $x = 3$, $x = 4$ about the x-axis.



Note: This is a Disk Method Problem!

So $V = \pi \int_3^4 \left(\sqrt{5x} + \sqrt{\frac{x}{5}} \right)^2 dx$

$$V = \pi \int_3^4 \left(5x + 2\sqrt{\frac{5x \cdot x}{5}} + \frac{x}{5} \right) dx = \pi \int_3^4 \left(\frac{26}{5}x + 2\sqrt{x^2} \right) dx$$

$$= \pi \int_3^4 \left(\frac{26}{5}x + 2x \right) dx = \pi \int_3^4 \frac{36}{5} x \, dx = \frac{36}{5}\pi \left. \frac{x^2}{2} \right]_3^4$$

$$= \frac{18}{5}\pi \left(4^2 - 3^2 \right) = \frac{18}{5}\pi (7) = \frac{126}{5}\pi$$