Lesson 14: Volume of Solids of Revolution

Let's recap some items from Lessons 12 and 13,

- Formulas

  For rotation around x-axis
  - Disk Method: $V = \pi \int_a^b [f(x)]^2 dx$
  - Washer Method: $V = \pi \int_a^b (R^2 - r^2) dx$

  For rotation around y-axis
  - Disk Method: $V = \pi \int_c^d [g(y)]^2 dy$
  - Washer Method: $V = \pi \int_c^d (R^2 - r^2) dy$

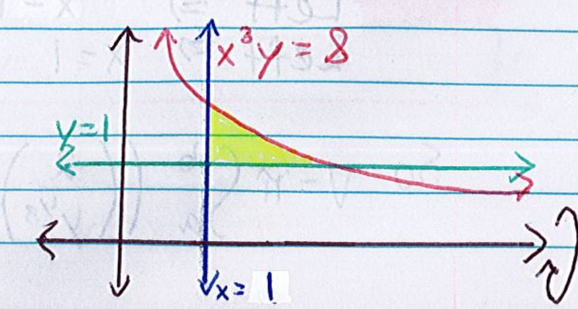
- When to apply Disk Method or Washer Method?
  - ↳ When the region "hugs" the axis of rotation ⇒ Disk Method
  - ↳ When there is a "gap" between the region and axis of rotation ⇒ Washer Method

In the previous two lessons, we looked at rotation around the x-axis or y-axis. Today, we are going to rotate about ANY arbitrary axis.
(We are going to limit ourselves to any vertical or horizontal line parallel to the x-axis or y-axis)

Example 1: Let R be the region of the xy-plane bounded above by the curves $x^3y = 8$, below by the line $y=1$, on the left by the line $x=1$. Find the volume of the solid obtained by rotating R around

ⓐ the x-axis

First draw the region (as shown on the right). From the graph, we can see this is WASHER problem.



Since this is rotation around x-axis, we need

Top $\Rightarrow x^3y=8 \longleftrightarrow y=8/x^3$

Bottom $\Rightarrow y=1 \longleftrightarrow y=1$

So $V=\pi\int_a^b \left(\left(\frac{8}{x^3}\right)^2-1^2\right)dx$

Now find a & b. Well we can see the smallest value of x is 1 so $a=1$. To find b, set $y=8/x^3$ equal to $y=1$.

$$1=\frac{8}{x^3} \Leftrightarrow x^3=8 \Leftrightarrow x=2$$

So $V=\pi\int_1^2\left(\frac{64}{x^6}-1\right)dx=\pi\int_1^2(64x^{-6}-1)dx$

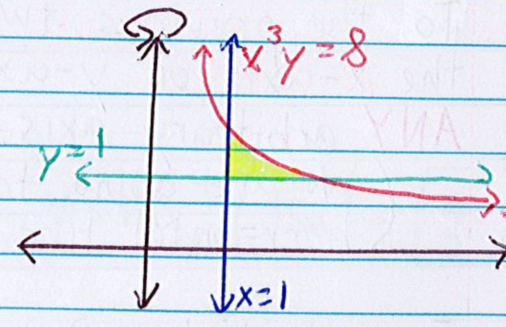
Now integrate.

$$V=\pi\left(\frac{64x^{-5}}{-5}-x\right)\Big]_1^2=\pi\left(\frac{-64}{5}\cdot\frac{1}{x^5}-x\right)\Big]_1^2$$

$$=\pi\left(\frac{-64}{5}\cdot\frac{1}{2^5}-2\right)-\pi\left(\frac{-64}{5}-1\right)=\frac{57}{5}\pi$$

(b) the y-axis

Good new is that we can use our picture from (a) again. Note this problem is also a WASHER problem.



Since this is rotation around y-axis, we need

Right $\Rightarrow x^3y=8 \longleftrightarrow x^3=\frac{8}{y} \leftrightarrow x=\sqrt[3]{\frac{8}{y}}=\frac{2}{y^{1/3}}$

Left $\Rightarrow x=1 \longleftrightarrow x=1$

So $V=\pi\int_a^b\left(\left(\frac{2}{y^{1/3}}\right)^2-1^2\right)dy$

Now find a & b. We can see the smallest value of y is 1 so a=0. To find b, set $x = \frac{2}{y^{1/3}}$ to $x=1$.

$$1 = \frac{2}{y^{1/3}} \Leftrightarrow y^{1/3} = 2 \Leftrightarrow y = 2^3 = 8$$

$$\text{So } V = \pi \int_1^8 \left(\frac{4}{y^{2/3}} - 1 \right) dy = \pi \int_1^8 \left(4y^{-2/3} - 1 \right) dy$$

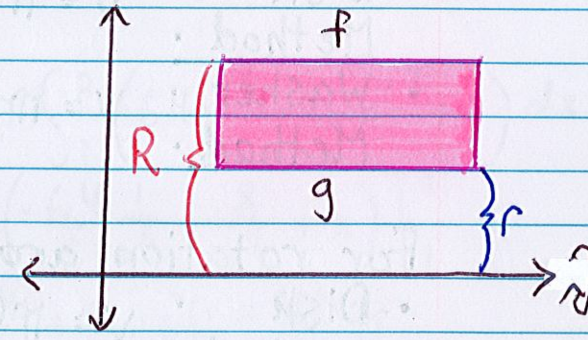
Now integrate

$$V = \pi \left(4 \cdot \frac{3}{1} y^{1/3} - y \right) \Big]_1^8 = \pi \left(12(8)^{1/3} - 8 \right) - \pi \left(12(1)^{1/3} - 1 \right)$$

$$= \pi (24 - 8 - 12 + 1) = 5\pi$$

<u>Let's Backtrack a Bit</u>

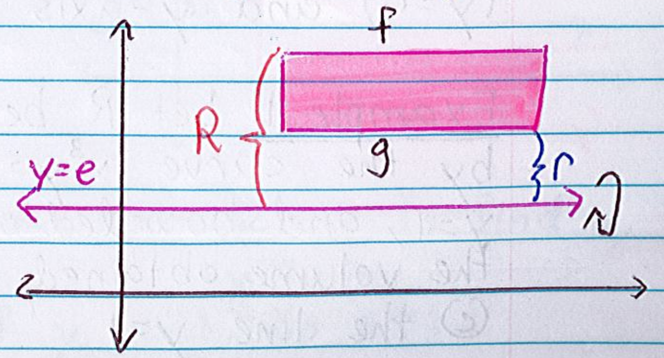
Remember when we first described Washers, we talked about farthest and closest.



Consider the case of x-axis rotation. In terms of distance,

- R is the length of Top Function away from x-axis i.e. $R = f$
- r is the length of Bottom Function away from x-axis i.e. $r = g$

So what is the distance between f (and g) and $y=e$?



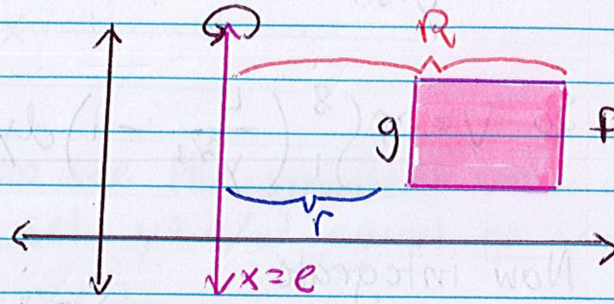
- Distance between f and $y=e$ is $R = f - e$
- Distance between g and $y=e$ is $r = g - e$

Note this formula is also true for the x-axis case because the x-axis is simply the line $y=0$.

When rotating around the line $y=e$, the same formulas for R and r apply.

So the distance between f (and g) and $x=e$ is:

- $R = f - e$
- $r = g - e$



Note that though we did all these calculations for the Washer Method, this also applies for the Disk Problems.

Rotation around any non-Axis Formulas

For rotation around the line $y=e$:

- Disk Method: $V = \pi \int_a^b [f(x) - e]^2 \, dx$
- Washer Method: $V = \pi \int_a^b \left[(R-e)^2 - (r-e)^2 \right] dx$

For rotation around the line $x=e$:

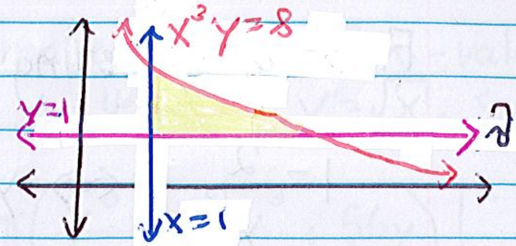
- Disk Method: $V = \pi \int_c^d [g(y) - e]^2 \, dy$
- Washer Method: $V = \pi \int_c^d \left[(R-e)^2 - (r-e)^2 \right] dy$

Note that these formulas work for the cases of x-axis ($y=0$) and y-axis ($x=0$).

Example 1: Let R be the region of the xy-plane bounded by the curve $x^3 y = 8$, bounded below by the line $y=1$, and bounded on the left by the line $x=1$. Find the volume obtained by rotating R around

(c) the line $y=1$

From the graph, we can see the region is "hugging" the line of rotation $y=1$. So this a DISK problem.



$$V=\pi\int_a^b \left(f(x)-1\right)^2 dx$$

Remember $x^3y=8 \Leftrightarrow y=\frac{8}{x^3}$. To find a, note that the smallest value of x is 1. So $a=1$. Find b set $y=1$ and $y=\frac{8}{x^3}$ equal.

$$1=\frac{8}{x^3} \Leftrightarrow x^3=8 \Leftrightarrow x=2$$

So our integral is $V=\pi\int_1^2 \left(\frac{8}{x^3}-1\right)^2 dx$

Now integrate

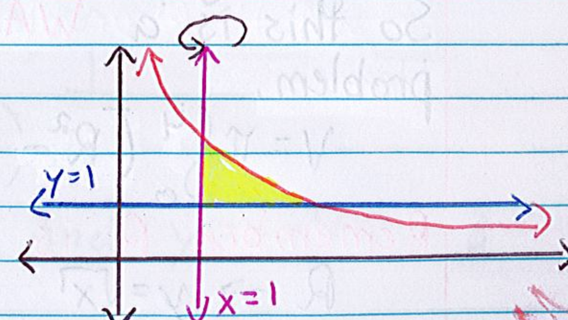
$$V=\pi\int_1^2\left(\frac{64}{x^6}-\frac{16}{x^3}+1\right)dx=\pi\int_1^2\left(64x^{-6}-16x^{-3}+1\right)dx$$

$$=\pi\left(\frac{64}{-5}x^{-5}-\frac{16}{-2}x^{-2}+x\right)\Big]_1^2=\pi\left(-\frac{64}{5}\cdot\frac{1}{x^5}+\frac{8}{x^2}+x\right)\Big]_1^2$$

$$=\pi\left(-\frac{64}{5}\cdot\frac{1}{2^5}+\frac{8}{2^2}+2\right)-\pi\left(-\frac{64}{5}+8+1\right)=\frac{41}{5}\pi$$

(d) the line $x=1$

From the graph, we can see the region is "hugging" the line of rotation $x=1$. So this a DISK problem.



$$V=\pi\int_a^b (g(y)-1)^2 dy$$

Remember $x^3y=8 \Leftrightarrow x^3=\frac{8}{y} \Leftrightarrow x=\frac{2}{y^{1/3}}$. To find a, note that the smallest value of y is 1. So $a=1$.

Find b by setting $x=1$ and $x=\frac{2}{y^{1/3}}$ equal.

$$1=\frac{2}{y^{1/3}} \iff y^{1/3}=2 \iff y=8$$

So our integral is $V=\pi\int_1^8 \left(\frac{2}{y^{1/3}}-1\right)^2 dy$

Now integrate

$$V=\pi\int_1^8\left(\frac{4}{y^{2/3}}-\frac{4}{y^{1/3}}+1\right)dy=\pi\int_1^8\left(4y^{-2/3}-4y^{-1/3}+1\right)dy$$

$$=\pi\left(\frac{4\cdot 3}{1}y^{1/3}-\frac{4\cdot 3}{2}y^{2/3}+y\right)\Big]_1^8=\pi\left(12y^{1/3}-6y^{2/3}+y\right)\Big]_1^8$$

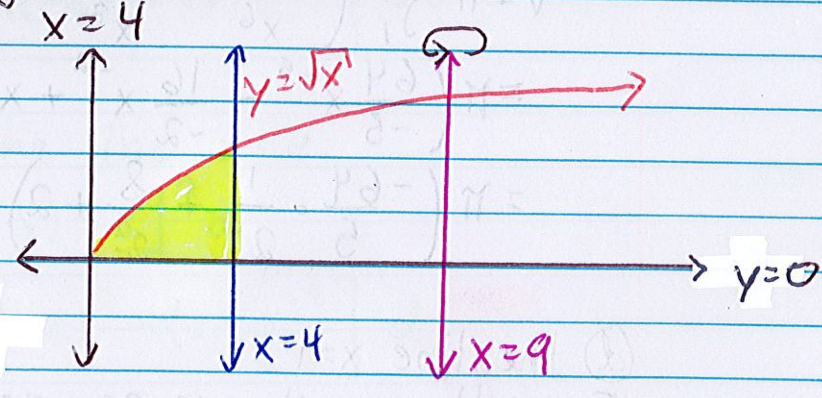
$$=\pi\left(12\cdot 8^{1/3}-6\cdot 8^{2/3}+8\right)-\pi(12-6+1)$$

$$=(8-7)\pi=\pi$$

Example 2: Find the volume of the solid generated by revolving the given region about the line $x=9$:

$$y=\sqrt{x},\quad y=0,\quad x=4$$

From the graph, we can see that the region and the line $x=9$ have a "gap." So this is a WASHER problem.



Hence

$$V=\pi\int_a^b\left[(R-9)^2-(r-9)^2\right]dy$$

Remember R is the farthest and r is the closest

$$R \rightarrow y=\sqrt{x} \longleftrightarrow x=y^2$$
$$r \rightarrow x=4 \longleftrightarrow x=4$$

So $V=\pi\int_a^b\left[(y^2-9)^2-(4-9)^2\right]dy$

$$=\pi\int_a^b\left[y^4-18y^2+81-25\right]dy=\pi\int_a^b\left[y^4-18y^2+56\right]dy$$

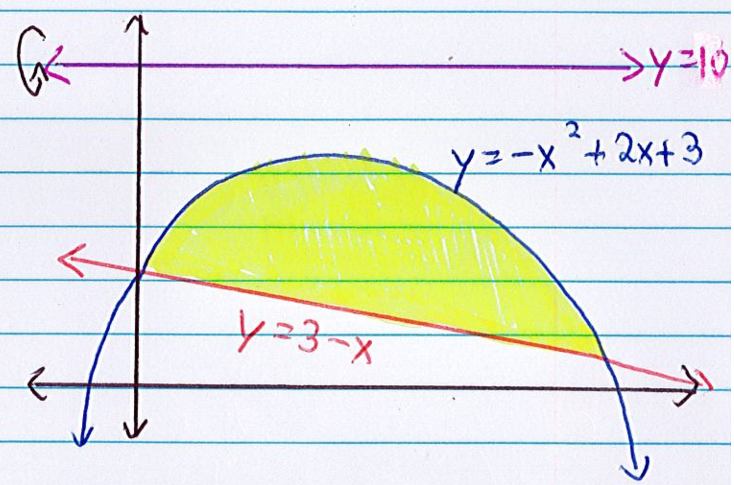
Now let's find a & b. Note the smallest value of y-values is 0. So $a=0$. To find b, plug $x=4$ into $y=\sqrt{x}$. So $b=2$. Hence

$$V=\pi\int_0^2 (y^4-18y^2+56)\,dy=\pi\left(\frac{y^5}{5}-\frac{18y^3}{3}+56y\right)\Big]_0^2$$

$$=\pi\left(\frac{2^5}{5}-6(2)^3+56(2)\right)=\frac{352}{5}\pi$$

Example 3: Find the volume of the solid generated by revolving the given region about the line $y=10$.

$$y=-x^2+2x+3 \quad \text{and} \quad y=3-x$$



From the graph, we can see that the region and the line $y=10$ have a gap. So this is a WASHER problem. Hence

$$V=\pi\int_a^b \left[(R-10)^2-(r-10)^2\right]dx$$

Remember R is the farthest and r is the closest

$R \to y=3-x$

$r \to y=-x^2+2x+3$

So $V=\pi\int_a^b \left[(3-x-10)^2-(-x^2+2x+3-10)^2\right]dx$

$$=\pi\int_a^b \left[(-7-x)^2-(-x^2+2x-7)^2\right]dx$$

$$=\pi\int_a^b (-x^4+4x^3-17x^2+42x)\,dx$$

To find a and b set $y=3-x$ and $y=-x^2+2x+3$ equal.

$$3-x=-x^2+2x+3$$
$$0=-x^2+3x$$
$$0=-x(x-3)$$
$$x=0,\ 3$$

$$V = \pi \int_0^3 \left(-x^4 + 4x^3 - 17x^2 + 42x\right) dx$$

$$= \pi \left(\frac{-x^5}{5} + \frac{4x^4}{4} - \frac{17x^3}{3} + \frac{42x^2}{2} \right) \Bigg]_0^3$$

$$= \pi \left(-\frac{3^5}{5} + (3)^4 - \frac{17}{3}(3)^3 + 21(3)^2 \right) = \frac{342}{5}\pi$$