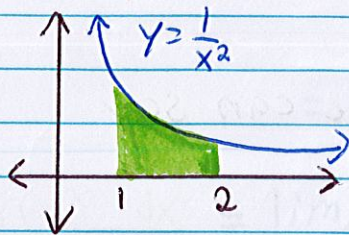


# Lesson 15: Improper Integrals

Suppose we want to find the area under the curve  $y = \frac{1}{x^2}$  with bounds  $x=1$  and  $x=2$ .



$$A = \int_1^2 \frac{1}{x^2} dx = \left. -\frac{1}{x} \right|_1^2 = -\frac{1}{2} + 1 = \frac{1}{2}$$

How about I change the bounds to be  $x=1$  and  $x=3$ ?

$$A = \int_1^3 \frac{1}{x^2} dx = \left. -\frac{1}{x} \right|_1^3 = -\frac{1}{3} + 1 = \frac{2}{3}$$

How about I change the bounds to be  $x=1$  and  $x=4$ ?

$$A = \int_1^4 \frac{1}{x^2} dx = \left. -\frac{1}{x} \right|_1^4 = -\frac{1}{4} + 1 = \frac{3}{4}$$

So note  $A$  gets closer and closer to 1 as the top bound gets larger. So we can say

$$\int_1^{\infty} \frac{1}{x^2} dx = 1$$

This integral on the left is what we can call an **improper integral**.

**Definition:** An improper integral is a definite integral  $\int_a^b f(x) dx$  such that the integrand  $f(x)$  is defined

on  $(a, b)$  but not necessarily at  $a$  or  $b$ .

ex.  $\int_a^{\infty} f(x) dx$  ;  $\int_{-\infty}^b f(x) dx$  ;  $\int_{-\infty}^{\infty} f(x) dx$

To compute such integrals, we need to bring back the notion of limits.



Definition: If  $f(x)$  approaches  $L$  as  $x$  approaches  $c$ , we say that the limit of  $f(x)$  as  $x$  approaches  $c$  is  $L$ .

$$\text{i.e. } \lim_{x \rightarrow c} f(x) = L.$$

Now with the definition of limits, we can say

$$\int_a^{\infty} f(x) dx = \lim_{N \rightarrow \infty} \int_a^N f(x) dx$$

$$\int_{-\infty}^b f(x) dx = \lim_{N \rightarrow -\infty} \int_N^b f(x) dx$$

If the value of an improper integral is finite #, we say that the integral converges, and if not the integral diverges.

Example 1: Evaluate

$$\begin{aligned} \text{a) } \int_1^{\infty} \frac{1}{x^2} dx &= \lim_{N \rightarrow \infty} \int_1^N \frac{1}{x^2} dx = \lim_{N \rightarrow \infty} \left( \frac{-1}{x} \right) \Big|_1^N \\ &= \lim_{N \rightarrow \infty} \left( \frac{-1}{N} + 1 \right) = 1 \end{aligned}$$

$$\begin{aligned} \text{b) } \int_0^{\infty} 5xe^{-x} dx &= \lim_{N \rightarrow \infty} \int_0^N 5xe^{-x} dx \quad \begin{array}{l} u=5x \quad dv=e^{-x} dx \\ du=5dx \quad v=-e^{-x} \end{array} \\ &= \lim_{N \rightarrow \infty} \left( -5xe^{-x} \Big|_0^N - \int_0^N 5(-e^{-x}) dx \right) \\ &= \lim_{N \rightarrow \infty} \left( -5xe^{-x} \Big|_0^N + \int_0^N 5e^{-x} dx \right) \\ &= \lim_{N \rightarrow \infty} \left( -5xe^{-x} \Big|_0^N - 5e^{-x} \Big|_0^N \right) \\ &= \lim_{N \rightarrow \infty} \left( -5Ne^{-N} + 0 - 5e^{-N} + 5 \right) \\ &= \lim_{N \rightarrow \infty} \left( \underbrace{\frac{-5N}{e^N}}_0 - \underbrace{5e^{-N}}_0 + 5 \right) = 5 \end{aligned}$$



$$\begin{aligned}
 \textcircled{c} \int_2^{\infty} \frac{dx}{x\sqrt{\ln x}} &= \lim_{N \rightarrow \infty} \int_2^N \frac{dx}{x\sqrt{\ln x}} \quad \begin{array}{l} u = \ln x \\ du = \frac{1}{x} dx \end{array} \quad \lim_{N \rightarrow \infty} \int \frac{du}{u^{1/2}} \\
 &= \lim_{N \rightarrow \infty} \int u^{-1/2} du = \lim_{N \rightarrow \infty} \frac{2}{1} u^{1/2} \\
 &= \lim_{N \rightarrow \infty} \left( 2\sqrt{\ln x} \right)_2^N = \lim_{N \rightarrow \infty} \left( 2\sqrt{\ln N} - 2\sqrt{\ln 2} \right) \\
 &= \infty \Rightarrow \text{diverges}
 \end{aligned}$$

$$\begin{aligned}
 \textcircled{d} \int_2^{\infty} \frac{dx}{x} &= \lim_{N \rightarrow \infty} \int_2^N \frac{dx}{x} = \lim_{N \rightarrow \infty} \left( \ln|x| \right)_2^N \\
 &= \lim_{N \rightarrow \infty} \left( \ln|N| - \ln(2) \right) \\
 &= \infty \Rightarrow \text{diverges}
 \end{aligned}$$

Now there is one more integral that is also considered improper.

- If  $f$  is continuous on  $[a, b)$  and discontinuous on  $b$ , then

$$\int_a^b f(x) dx = \lim_{N \rightarrow b^-} \int_a^N f(x) dx$$

- If  $f$  is continuous on  $(a, b]$  and discontinuous on  $a$ , then

$$\int_a^b f(x) dx = \lim_{N \rightarrow a^+} \int_N^b f(x) dx$$

Example 2: Evaluate

$$\textcircled{a} \int_0^{\pi/2} \tan x dx$$

Note that  $\tan(x)$  @  $x = \pi/2$  is undefined.

$$= \lim_{N \rightarrow \pi/2^-} \int_0^N \frac{\sin x}{\cos x} dx \quad \begin{array}{l} u = \cos x \\ du = -\sin x \end{array}$$

$$= \lim_{N \rightarrow \pi/2^-} \left( -\frac{du}{u} \right) = \lim_{N \rightarrow \pi/2^-} \left( -\ln|u| \right)$$

$$= \lim_{N \rightarrow \pi/2^-} \left( -\ln|\cos x| \right)_0^N = \lim_{N \rightarrow \pi/2^-} \left( -\ln|\cos N| + \ln|\cos(0)| \right)$$



$$\begin{aligned}
 &= \lim_{N \rightarrow \pi/2^-} \left( -\ln |\cos(x)| + \ln |1| \right) \\
 &= -\ln \left| \cos \left( \frac{\pi}{2} \right) \right| + 0 = -\ln |0| = -\infty \Rightarrow \text{diverges}
 \end{aligned}$$

⑥  $\int_{-2}^{14} \frac{dx}{4\sqrt{x+2}}$  Note that  $\frac{1}{4\sqrt{x+2}}$  is undefined at  $x=-2$

$$\begin{aligned}
 &= \lim_{N \rightarrow -2^+} \int_N^{14} (x+2)^{-1/4} dx = \lim_{N \rightarrow -2^+} \left( \frac{4}{3} (x+2)^{3/4} \right) \Big|_N^{14} \\
 &= \lim_{N \rightarrow -2^+} \left( \frac{4}{3} (14+2)^{3/4} - \frac{4}{3} (N+2)^{3/4} \right) \\
 &= \lim_{N \rightarrow -2^+} \left( \frac{32}{3} - \frac{4}{3} (N+2)^{3/4} \right) = \frac{32}{3} - \frac{4}{3} (0)^{3/4} = \frac{32}{3}
 \end{aligned}$$