Lesson 17: Geometric Series & Convergence

Recall from Last Class: If $0<|r|<1$, then
$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$$

From the Problem Set (posted online),

Example 1: How much should you invest today at an annual interest rate of 5% compounded continuously so that, starting next year, you can make annual withdrawals of \$2,000 in perpetuity

Interest compounded continuously $\Rightarrow A = Pe^{rt} = Pe^{0.05t}$

Since we want to make a withdrawal of \$2,000 every year, we will initially just consider the first few years and see if a pattern emerges:

- How much must we invest today P_1 so that we can withdrawal \$2000 one year from now?
$$2000 = P_1 \exp[0.05(1)]$$
$$P_1 = 2000e^{-0.05}$$

- How much must we invest today P_2 so that we can withdrawal \$2000 two years from now?
$$2000 = P_2 \exp[0.05(2)]$$
$$2000 = P_2 e^{0.01}$$
$$P_2 = 2000e^{-0.01}$$

- This pattern continues, we must invest
$$P_3 = 2000\exp[-0.05(3)] \text{ in } 3 \text{ yrs}$$
$$P_4 = 2000\exp[-0.05(4)] \text{ in } 4 \text{ yrs}$$
$$\vdots \qquad\qquad \vdots$$

$$\text{Total} = 2000\exp[-0.05(1)] + 2000\exp[-0.05(2)] + 2000\exp[-0.05(3)] + 2000\exp[-0.05(4)] + \ldots$$

Now let's write this as a sum.

$$\text{Total} = 2000\exp[-0.05]\left(1 + \exp[-0.05] + \exp[-0.05(2)] + \ldots\right)$$

$$= 2000\exp[-0.05] \cdot \sum_{n=0}^{\infty} (\exp[-0.05])^n$$

Now apply the Geometric Series Formula

$$\text{Total} = 2000\exp[-0.05] \cdot \frac{1}{1-\exp[-0.05]}$$

$$\approx \$39,008$$

Example 2: A patient is given an injection of 25 units of a drug every 24 hours. The drug is eliminated exponentially so that the fraction that remains after t days is given by $f(t) = e^{-t/3}$. If the treatment is continued indefinitely, approximately how many units will eventually be in the bloodstream just prior to an injection?

Since each injection is 25 units and the fraction that remains after t days from a single injection is $25e^{-t/3}$

$$\text{Total Amount in Body} = 25e^{-1/3} + 25e^{-2/3} + 25e^{-3/3} + \ldots$$

$$= 25e^{-1/3}\left(1 + e^{-1/3} + e^{-2/3} + \ldots\right)$$

$$= 25e^{-1/3} \sum_{n=0}^{\infty} e^{-n/3}$$

$$= 25e^{-1/3} \sum_{n=0}^{\infty} (e^{-1/3})^n$$

Now apply the Geometric Series Formula

$$\text{Total Amount in Body} = 25e^{-1/3} \cdot \frac{1}{1-e^{-1/3}}$$

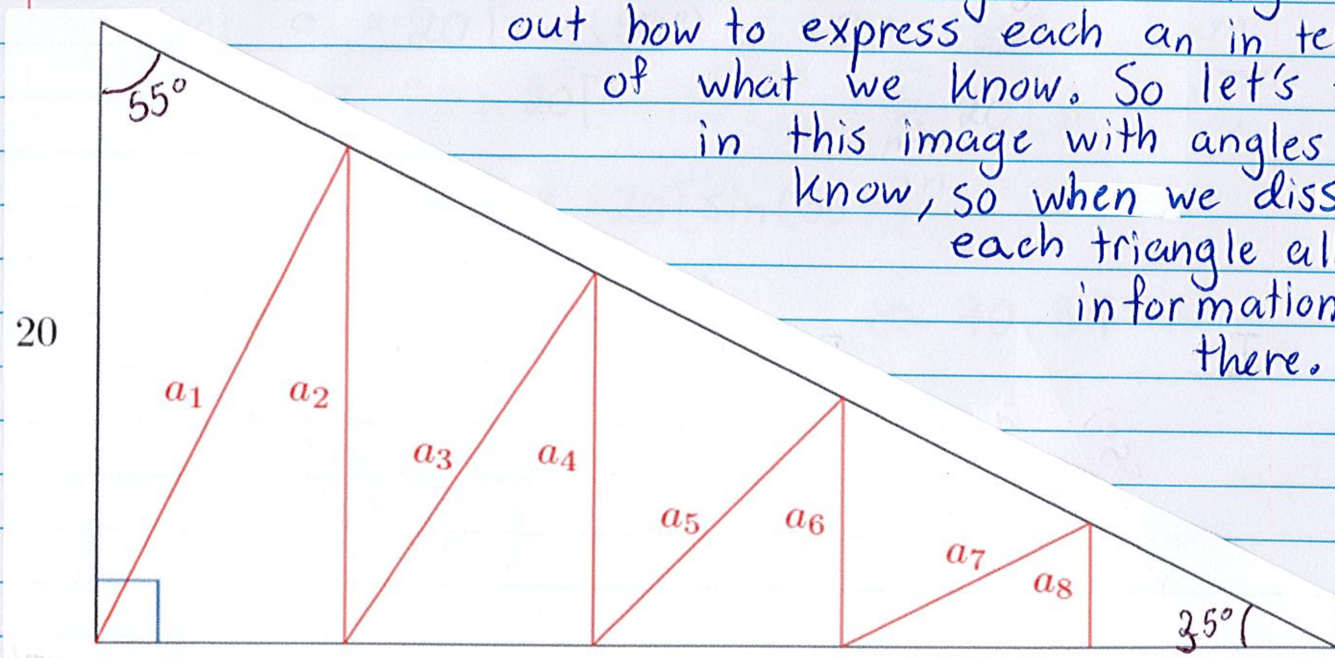
$$\approx 63.2 \text{ units}$$

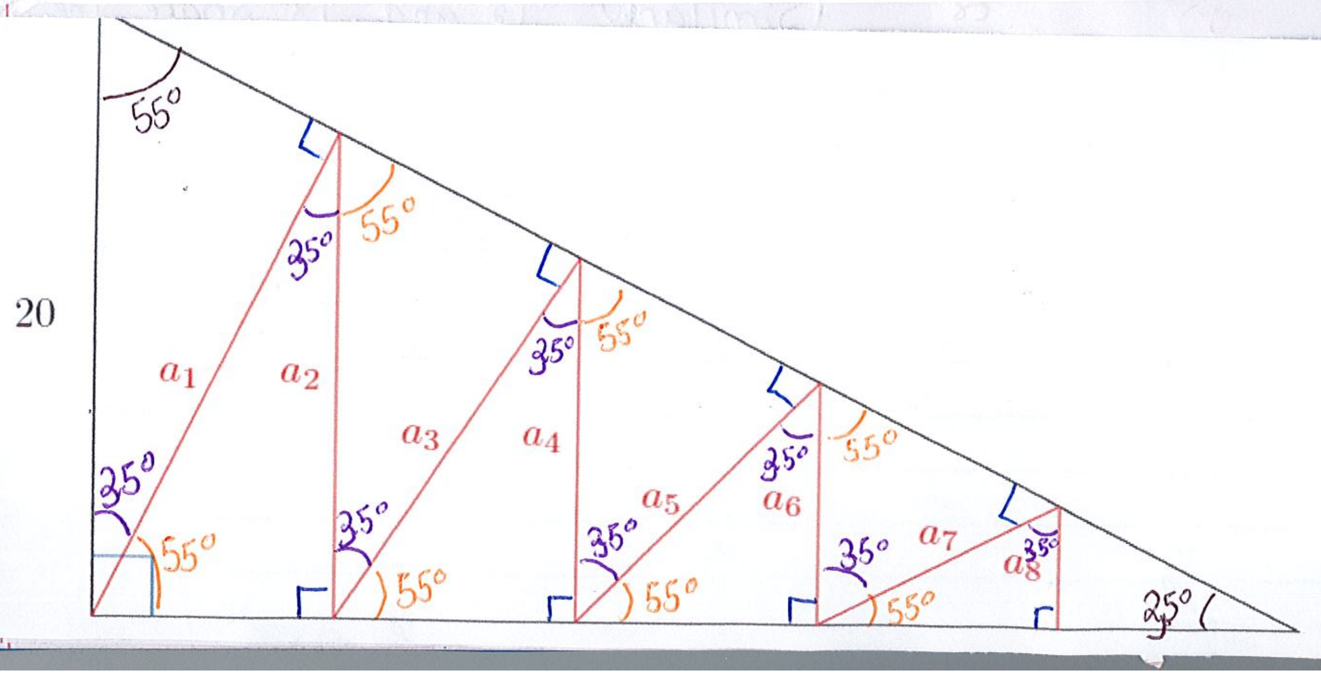
Example 3: A series of line segments are drawn inside a right triangle as follows:

1. An altitude is drawn from the right angle of the triangle.
2. In the new smaller right triangle formed that contains the smallest angle of the original triangle, another altitude is draw from the right angle of that triangle.
3. The process continues indefinitely, always moving toward the smallest angle of the original triangle.

Find the sum of the length of all these line segments if the original triangle has an angle of 55 degrees and the side adjacent to 55 degrees angle has length 20.

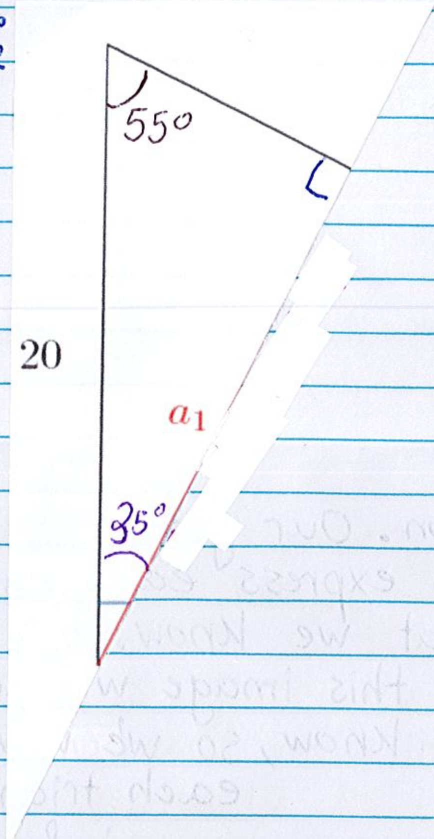
First let's illustrate this situation. Our goal is to figure out how to express each a_n in terms of what we know. So let's fill in this image with angles we know, so when we dissect each triangle all the information is there.





Note in each triangle we are looking for a_i.

T_1:



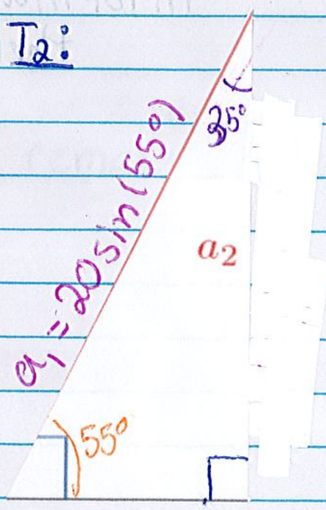
From the triangle, we have

$$\sin(55°) = \frac{a_1}{20}$$

$$a_1 = 20\sin(55°)$$

(Note T_1 and T_2 share the side a_1. Hence we will use that value in T_2.)

T_2:



From the triangle, we have

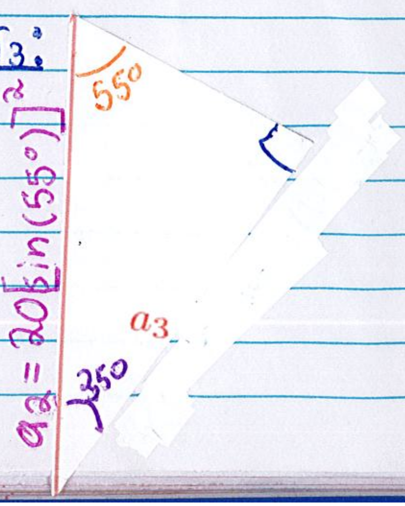
$$\sin(55°) = \frac{a_2}{20\sin(55°)}$$

$$a_2 = 20\sin(55°)\sin(55°)$$

$$= 20\left[\sin(55°)\right]^2$$

(Similarly, T_2 and T_3 share the side a_2)

T_3:



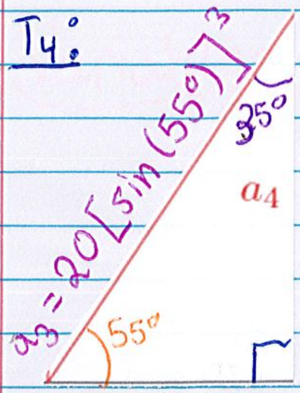
From the triangle, we have

$$\sin(55°) = \frac{a_3}{20\left[\sin(55°)\right]^2}$$

$$a_3 = 20\sin(55°)\left[\sin(55°)\right]^2$$

$$= 20\left[\sin(55°)\right]^3$$

T4:



From the triangle,

$$\sin(55°) = \frac{a_4}{20[\sin(55°)]^3}$$

$$a_4 = 20[\sin(55°)]^4$$

Hence $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} 20[\sin(55°)]^n$

Note the sum starts with $n=1$. So let's add and subtract $a_0 = 20[\sin(55°)]^0 = 20$ to the sum.

$$\sum_{n=1}^{\infty} a_n = -20 + 20[\sin(55°)]^0 + \sum_{n=1}^{\infty} 20[\sin(55°)]^n$$

$$= -20 + \sum_{n=0}^{\infty} 20[\sin(55°)]^n$$

$$= -20 + \frac{20}{1-\sin(55°)} \approx 90.59$$