Lesson 19: Partial Derivatives

A partial derivative is a derivative where we hold some variables constant.

Let's think about a function of one variable.
ex. $f(x) = x^2 \Rightarrow f'(x) = 2x$

But what about a function of two variables?
$$f(x,y) = x^2 + y^3$$
We find its partial derivative with respect to x by treating y as a constant.
$$f_x = 2x + 0 = 2x$$

To find the partial derivative with respect to y, we treat x as a constant.
$$f_y = 0 + 3y^2 = 3y^2$$

Definition: • The (first) partial derivative f_x describes the rate of change of f as x changes, where y remains constant. i.e. Find the derivative with respect to x, where we treat y as a constant

• The (first) partial derivative f_y describes the rate of change of f as y changes, where x remains constant. i.e. Find the derivative with respect to y, where we treat x as a constant.

Example 1: Compute the first order partial derivatives
ⓐ $f(x,y) = x^3 + 3xy$

First order partials $\Rightarrow$ We need to find f_x and f_y.

First find f_x. i.e. Find the derivative w/ respect to x and treat y as a constant.
$$f = x^3 + (3y)x$$
$$f_x = 3x^2 + 3y$$

Next find f_y. i.e. Find the derivative w/ respect to y and treat x as a constant.

$$f = x^3 + (3x)y$$
$$f_y = 0 + 3x \quad = 3x$$

Chain Rule Problem

ⓑ $f(x,y) = \ln(x+2y)$

First find f_x. i.e. Find the derivative w/ respect to x and treat y as a constant.

$$f_x = \frac{1}{x+2y} \cdot \frac{d}{dx}(x+2y) = \frac{1}{x+2y} \cdot (1+0) = \frac{1}{x+2y}$$

Next find f_y. i.e. Find the derivative w/ respect to y and treat x as a constant.

$$f_y = \frac{1}{x+2y} \cdot \frac{d}{dy}(x+2y) = \frac{1}{x+2y} \cdot (0+2) = \frac{2}{x+2y}$$

ⓒ $f(x,y) = \frac{9xy}{\sqrt{y-1}}$

First find f_x. i.e. Find the derivative w/ respect to x and treat y as a constant.

$$f(x,y) = \frac{9y}{\sqrt{y-1}}(x)$$
$$f_x = \frac{9y}{\sqrt{y-1}} \cdot \frac{d}{dx}(x) = \frac{9y}{\sqrt{y-1}}$$

Next find f_y. i.e. Find the derivative w/ respect to y and treat x as a constant.

$$f(x,y) = 9x\left(\frac{y}{\sqrt{y-1}}\right)$$

Apply Quotient Rule

$$f_y = 9x \cdot \frac{d}{dy}\left(\frac{y}{\sqrt{y-1}}\right) = 9x\left(\frac{1 \cdot \sqrt{y-1} - y \cdot \frac{1}{2}(y-1)^{-1/2}}{(\sqrt{y-1})^2}\right)$$
$$= 9x\left(\frac{\sqrt{y-1} - \frac{1}{2}y(y-1)^{-1/2}}{y-1}\right)$$

Example 2: Evaluate the partial derivatives $f_x(x,y)$ and $f_y(x,y)$ at the given point $P_0(x_0,y_0)$.

$$f(x,y) = x^3y^2 + 6x^2 \; ; \; P_0(1,-1)$$

First find f_x, i.e. Find the derivative w/ respect to x and treat y as a constant.

$$f_x = 3x^2y^2 + 12x^2$$

Plug $(1,-1)$ into f_x.

$$f_x(1,-1) = 3(1)^2(-1)^2 + 12(1)^2 = 15$$

Next find f_y, i.e. Find the derivative w/ respect to y and treat x as a constant.

$$f_y = x^3 \cdot 2y = 2x^3y$$

Plug $(1,-1)$ into f_y.

$$f_y(1,-1) = 2(1)^3(-1) = -2$$