# Lesson 1A: Integration By Substitution

Kinda of the Integration Version of the Chain Rule.
Also referred to as "Change of Variables"

Example 1: Find $\int 2x\cos(x^2)\,dx$

Idea to solve Example 1 is to Undo "chain rule."
First determine if you have a function within a function.
In this case,

$$\cos(x^2)$$

where $\cos(x)$ is the outer function, and
$x^2$ is the inner function.

What's the derivative of the inner function?

$$2x$$

Do you see $2x$ in the integrand?

YES!!!

Let's recall chain rule: $y' = f'(g(x)) \cdot g'(x)$ for $y = f(g(x))$
So from our example $y' = \cos(x^2) \cdot 2x$
i.e. $f(x) =$ $\qquad$ $g(x) = x^2$
$f'(x) = \cos x$ $\qquad$ $g'(x) = 2x$

So what's $f(x)$?

$$f(x) = \int f'(x)\,dx = \int \cos x\,dx = \sin x$$

Hence $y = \sin(x^2)$.
i.e. $\int 2x\cos(x^2)\,dx = \sin(x^2) + C$

## How to fast track this method?

Do a change of variable using the inner function.

## Solution to Example 1:

$$\int \cos(\underbrace{x^2}_{u})\,\underbrace{2x\,dx}_{du} \qquad \frac{u = x^2}{du = 2x\,dx} \qquad \int \cos(u)\,du$$

$$= \sin(u) + C$$

(But if we start w/. a function of x we want to end w/ a function of x. So sub back $u = x^2$.)

$$= \sin(x^2) + C$$

Note we may not see what du equals in our integrand so you may have to do some equation manipulation.

Example 2: Compute the following integrals.

ⓐ $\int \sqrt{4x+1}\,dx \qquad \frac{u = 4x+1}{du = 4\,dx} \qquad \int u^{1/2}\frac{du}{4} = \frac{1}{4}\int u^{1/2}\,du$

$\frac{du}{4} = dx$

$$= \frac{1}{4}\cdot\frac{2}{3}u^{3/2} + C = \frac{1}{6}(4x+1)^{3/2} + C$$

HW 1A #4 ⓑ $\int 5e^{\tan(14x)}\sec^2(14x)\,dx \qquad \frac{u = \tan(14x)}{du = \sec^2(14x)\cdot 14\,dx}$

$\frac{du}{14\sec^2(14x)} = dx$

$$= \int 5u\,\cancel{\sec^2(14x)}\cdot\frac{du}{14\,\cancel{\sec^2(14x)}} = \frac{5}{14}\int u\,du = \frac{5}{14}\cdot\frac{u^2}{2} + C$$

$$= \frac{5}{28}(\tan(14x))^2 + C$$

ⓒ $\int \frac{10x^3 - 5x}{\sqrt{x^4 - x^2 + 6}}\,dx$ (When you have fractions, it helps to rewrite to a product)

$$= \int (10x^3 - 5x)(x^4 - x^2 + 6)^{-1/2}\,dx \qquad u = x^4 - x^2 + 6$$

$du = 4x^3 - 2x\,dx$

$\frac{du}{4x^3 - 2x} = dx$

$$= \int (10x^3 - 5x)\, u^{-1/2} \frac{du}{4x^3 - 2x} = \int \frac{5\cancel{(2x^3-1)}}{2\cancel{(2x^3-1)}} u^{-1/2}\, du$$

$$= \frac{5}{2} \int u^{-1/2}\, du = \frac{5}{2} \cdot \frac{2}{1} u^{1/2} + C = 5(x^4 - x^2 + 6)^{1/2} + C$$

HW 1A #6 Example 3: Find the function $f(x)$ whose tangent line has the slope $\frac{(1+\sqrt{x})^{1/2}}{4\sqrt{x}}$ for any $x \neq 0$ and whose graph passes through the point $(9, 5/3)$.

$$f'(x) = \frac{(1+\sqrt{x})^{1/2}}{4\sqrt{x}}$$

So $f(x) = \int f'(x)\, dx = \int (1+\sqrt{x})^{1/2} \cdot \frac{1}{4} \cdot \frac{1}{\sqrt{x}}\, dx$

$u = 1 + \sqrt{x}$

$du = \frac{1}{2\sqrt{x}}\, dx$

$2du = \frac{1}{\sqrt{x}}\, dx$

$$= \int u^{1/2} \cdot \frac{1}{4} \cdot 2du = \frac{1}{2} \int u^{1/2} du = \frac{1}{2} \cdot \frac{2}{3} u^{3/2} + C$$

$$= \frac{1}{3}(1+\sqrt{x})^{3/2} + C$$

Now we need to find C. (w/ $(9, 5/3)$)

$$\frac{5}{3} = f(9) = \frac{1}{3}(1+\sqrt{9})^{3/2} + C$$

$$\frac{5}{3} = \frac{8}{3} + C$$

$$-1 = \frac{-3}{3} = C$$

So $f(x) = \frac{1}{3}(1+\sqrt{x})^{3/2} - 1$

Moral: You must eliminate all instances of the original variable x when you change the variable.