Lesson 1B: Integration By Substitution

Last time, when $\int f(x)dx = \int g(u(x)) \cdot u'(x)\,dx$

1. Choose $u(x)$
2. Then $du = u'(x)dx$

To produce a new integral $\int f(x)dx = \int g(u)du$. Then intergate and plug back $u(x)$ to get our answer.

Today, we will do examples where we need to be more clever.

Example 1: Compute

(a) $\int x^3\sqrt{1-x^2}\,dx$ $\quad \dfrac{u = 1-x^2}{du = -2x\,dx}$ $\quad \dfrac{du}{-2x} = dx$ $\quad \int x^3 u^{1/2}\dfrac{du}{-2x} = \dfrac{-1}{2}\int x^2 u^{1/2}du$

~~Issue~~: Our integral as x and u! Can I rewrite x^2 into some relation to u? Yes!

$u = 1-x^2 \iff x^2 = 1-u$

$= -\dfrac{1}{2}\int(1-u)u^{1/2}du = -\dfrac{1}{2}\int u^{1/2} - u^{3/2}\,du$

$= -\dfrac{1}{2}\left(\dfrac{2}{3}u^{3/2} - \dfrac{2}{5}u^{5/2}\right) + C = -\dfrac{1}{3}(1-x^2)^{3/2} + \dfrac{1}{5}(1-x^2)^{5/2} + C$

(b) $\int \dfrac{x}{\sqrt{x+2}}dx$ $\quad \dfrac{u = x+2}{du = dx} \iff x = u-2$ $\quad \int \dfrac{u-2}{u^{1/2}}du$

$= \int u^{1/2} - 2u^{-1/2}\,du = \dfrac{2}{3}u^{3/2} - 2\cdot\dfrac{2}{1}u^{1/2} + C$

$= \dfrac{2}{3}(x+2)^{3/2} - 4(x+2)^{1/2} + C$

Example 2: Compute

(a) $\int_2^4 x\sin(x^2)\,dx$ $\quad \dfrac{u=x^2}{du=2x\,dx}$ $\quad \int \cancel{x}\sin(u)\,\dfrac{du}{2x} = \dfrac{1}{2}\int \sin(u)\,du$

$\dfrac{du}{2x} = dx$

$= -\dfrac{1}{2}\cos(u) = -\dfrac{1}{2}\cos(x^2)\Big]_2^4$

$= -\dfrac{1}{2}\cos(16) + \dfrac{1}{2}\cos(4)$

(b) $\int_{1/3}^{1/2} \dfrac{e^{1/x}}{x^2}\,dx = \int_{1/3}^{1/2} e^{1/x}\cdot\dfrac{1}{x^2}\,dx$ $\quad \dfrac{u=1/x}{du=-1/x^2\,dx}$ $\quad \int -e^u\,du$

$-du = 1/x^2\,dx$

$= -e^u = -e^{1/x}\Big]_{1/3}^{1/2} = -e^2 + e^3$

Example 3: Find the area of the region R that lies under the curve $y = \dfrac{3}{\sqrt{9-2x}}$ over the interval $-8 \le x \le 0$.

Area $= \int_{-8}^{0} \dfrac{3}{\sqrt{9-2x}}\,dx$ $\quad \dfrac{u=9-2x}{du=-2\,dx}$ $\quad \int \dfrac{3}{u^{1/2}}\cdot\dfrac{du}{-2} = -\dfrac{3}{2}\int u^{-1/2}\,du$

$\dfrac{du}{-2} = dx$

$= -\dfrac{3}{2}\cdot\dfrac{2}{1}u^{1/2} = -3(9-2x)^{1/2}\Big]_{-8}^{0} = -3(9)^{1/2} + 3(9+16)^{1/2}$

$= -3\cdot 3 + 3\cdot 5 = -9 + 15 = 6$