Lesson 21: Differentials of Multivariable Functions

Recall that if $y = f(x)$ then $\frac{dy}{dx} = f'(x)$. i.e. $dy = f'(x)dx$

Also recall that the derivative is the slope of the tangent line. So

$$f'(x) \approx \text{Slope} = \frac{\Delta y}{\Delta x} \qquad \text{i.e. } \Delta y \approx f'(x)\Delta x$$

By extension, if $f(x,y)$ is a function of two variables, we can consider small changes in x and y as Δx and Δy, respectively, and express dz and Δz in terms of its partials

$$dz = f_x(x,y)\,dx + f_y(x,y)\,dy$$

and $\Delta z \approx (x\text{-Slope})\Delta x + (y\text{-Slope})\Delta y$

$$\Delta z \approx \frac{dz}{dx} \cdot \Delta x + \frac{dz}{dy}\Delta y$$

Note: $dy \neq \Delta y$ because dy represents an infinitesimal change in y while Δy is how y changes over a small finite interval. The same is true for Δx and Δz.

From the Problem Set (posted online),

Example 1: Use increments to estimate the change in z at $(x,y) = (0,0)$ if $\frac{dz}{dx} = -8x + 6$ and $\frac{dz}{dy} = 6y + 4$. The change in x is 0.3 and the change in y is 0.2.

From the problem, $\Delta x = 0.3$ and $\Delta y = 0.2$. So,

$$\begin{aligned}\Delta z &= \left.\frac{dz}{dx}\right|_{(0,0)} \Delta x + \left.\frac{dz}{dy}\right|_{(0,0)} \Delta y \\ &= (-8(0)+6)\cdot(0.3) + (6(0)+4)\cdot(0.2) \\ &= 6(0.3) + 4(0.2) \\ &= 1.8 + 0.8 \\ &= 2.6\end{aligned}$$

Example 2: Find $z=f(x,y)$ and use the total differential to approximate the quantity

$$\sqrt{(5.3)^2+(7.45)^2}-\sqrt{5^2+7^2}$$

Let's make $f(x,y)=\sqrt{x^2+y^2}=(x^2+y^2)^{1/2}$. So $x=5$, $y=7$, $\boxed{\Delta x=0.3}$, and $\boxed{\Delta y=0.45}$. Hence we can rewrite that given equation as

$$\Delta z = f(5.3, 7.45) - f(5,7)$$

Using the formula,

$$\Delta z \approx \left.\frac{dz}{dx}\right|_{(5,7)} \Delta x + \left.\frac{dz}{dy}\right|_{(5,7)} \Delta y$$

to approximate Δz.

$$\frac{dz}{dx} = \frac{d}{dx}\left((x^2+y^2)^{1/2}\right) = \frac{1}{2}(x^2+y^2)^{-1/2}\cdot 2x = \frac{x}{\sqrt{x^2+y^2}}$$

$$\left.\frac{dz}{dx}\right|_{(5,7)} = \frac{5}{\sqrt{5^2+7^2}} = \boxed{\frac{5}{\sqrt{74}}}$$

$$\frac{dz}{dy} = \frac{d}{dy}\left((x^2+y^2)^{1/2}\right) = \frac{1}{2}(x^2+y^2)^{-1/2}\cdot 2y = \frac{y}{\sqrt{x^2+y^2}}$$

$$\left.\frac{dz}{dy}\right|_{(5,7)} = \frac{7}{\sqrt{5^2+7^2}} = \boxed{\frac{7}{\sqrt{74}}}$$

Plugging in all the **pink boxes** into the formula,

$$\Delta z = \left(\frac{5}{\sqrt{74}}\right)0.3 + \left(\frac{7}{\sqrt{74}}\right)0.45 \approx 0.5406$$

Example 3: The specific gravity of an object with density greater than that of water can be determined by using the formula $S=\dfrac{A}{A-W}$ where A and W are the weights of the object in air and in water, respectively. If the measurements of an object are $A=2.2$ lb and $W=2$ lb with max errors of 0.02 lb and 0.02 lb, respectively.

ⓐ Find the approximate maximum error in calculating S.

From the problem, $A = 2.2$, $W = 2$, $\Delta A = \pm 0.02$ and $\Delta W = \pm 0.02$.

$$S_A = \frac{1(A-W) - A(1)}{(A-W)^2} = \frac{A-W-A}{(A-W)^2} = \frac{-W}{(A-W)^2}$$

$$S_A(2.2, 2) = \frac{-2}{(2.2-2)^2} = -50$$

Note $S = A(A-W)^{-1}$

$$S_W = -A(A-W)^{-2} \cdot (-1) = \frac{A}{(A-W)^2}$$

$$S_W(2.2, 2) = \frac{2.2}{(2.2-2)^2} = 50$$

$$\text{So } \Delta S = S_A(2.2, 2)\,\Delta A + S_W(2.2, 2)\,\Delta W$$
$$= -50\,\Delta A + 55\,\Delta W$$

Note that there is 4 combination of values of ΔA and ΔW, because of the $\pm$. We need to plug in each combination and pick the largest number. This number will be our maximum error.

- When $\Delta A = 0.02$, $\Delta W = 0.02$
  $\Delta S = -50(0.02) + 55(0.02) = 0.1$
- When $\Delta A = 0.02$, $\Delta W = -0.02$
  $\Delta S = -50(0.02) + 55(-0.02) = -2.1$
- When $\Delta A = -0.02$, $\Delta W = 0.02$
  $\Delta S = -50(-0.02) + 55(0.02) = 2.1$
- When $\Delta A = -0.02$, $\Delta W = -0.02$
  $\Delta S = -50(-0.02) + 55(-0.02) = -0.1$

Final Answer is maximum error is 2.1

(b) Find the approximate relative percentage error in calculating S.

$$\text{Relative Percentage Error} = \frac{\Delta S}{S} \doteq \frac{2.1}{\frac{2.2}{2.2-2}} = \frac{2.1}{1} \cdot \frac{0.2}{2.2} = 0.1909 \Rightarrow 19.09\%$$

Example 4: Hot chocolate sales (in gallons) are predicted by $f(x,y) = 1.7\, x^{1/2} y^{-1}$ where y is the temperature (in °C) and x is the amount of snow on the ground (in inches). If the temperature rises from 1.6° to 1.7° and the amount of snow falls from 2 inches to 1.2 inches, use differentials to estimate the change in hot chocolate sales.

ⓐ The change in hot chocolate sales due to the change in the amount of snow is:

i.e. Find $f_x(2, 1.6)\,\Delta x$.

$$f_x(x,y) = 1.7(\tfrac{1}{2})x^{-1/2}y^{-1} = \frac{0.85}{\sqrt{x}\,y} = \frac{0.85}{y\sqrt{x}}$$

$$f_x(2,1.6) = \frac{0.85}{1.6\sqrt{2}}$$

$$\Delta x = 1.2 - 2 = -0.8$$

Hence $f_x(2,1.6)\,\Delta x = \dfrac{0.85}{1.6\sqrt{2}} \cdot (-0.8) \approx -0.301$

ⓑ The change in hot chocolate sales due to the change in the temperature is:

i.e. Find $f_y(2, 1.6)\,\Delta y$

$$f_y(x,y) = -1.7\, x^{1/2} y^{-2} = \frac{-1.7\sqrt{x}}{y^2}$$

$$f_y(2,1.6) = \frac{-1.7\sqrt{2}}{(1.6)^2}$$

$$\Delta y = 1.7 - 1.6 = 0.1$$

Hence $f_y(2,1.6)\,\Delta y = \dfrac{-1.7\sqrt{2}}{(1.6)^2} \cdot (0.1) \approx -0.094$

(c) The total change in hot chocolate sales is:

$$\Delta f = f_x(2, 1.6)\Delta x + f_y(2, 1.6)\Delta y \approx -0.394$$

Example 5: A soft drink can is a cylinder h cm tall with radius r cm. It's volume is given by the formula $V(r,h) = \pi r^2 h$. A particular can is 14 cm tall and has a radius of 4 cm. If the height is increased by 1.2 cm, use calculus to estimate the change in the radius needed so that the volume stays the same.

i.e. We want Δr when $\Delta V = 0$.

From the problem, $h = 14$, $r = 4$, and $\Delta h = 1.2$

$V_r(r,h) = 2\pi r h$ $\qquad$ $V_h(r,h) = \pi r^2$

$V_r(4,14) = 112\pi$ $\qquad$ $V_h(4,14) = 16\pi$

Hence
$$\Delta V = V_r(4,14)\Delta r + V_h(4,14)\Delta h$$
$$= 112\pi \Delta r + 16\pi(1.2)$$

Set $\Delta V = 0$ and solve for Δr.

$$0 = 112\pi \Delta r + 19.2\pi$$
$$-19.2\pi = 112\pi \Delta r$$
$$\Delta r = -\frac{19.2\pi}{112\pi} \approx -0.1714$$