# Lesson 22: Chain Rule for Multivariable Functions

Recall the chain rule for single variable functions

Let $h(x) = f(g(x))$. Then $h'(x) = f'(g(x))g'(x)$.

Another notation of this chain rule is as follows:

Let $u = g(x)$ and $y = f(u)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

We can extend this notation for two variable functions. Assume each function in the following statement are differentiable.

Let $z = f(x,y)$, $x = g(t)$, $y = h(t)$, then

$$\frac{dz}{dt} = \frac{dz}{dx} \cdot \frac{dx}{dt} + \frac{dz}{dy} \cdot \frac{dy}{dt}$$

Example 1: Find $dz/dt$. Express your answer in terms $x, y,$ and $t$.

$$z = \sin(x^2+y^2) \ ; \quad x = 8t^2+3 \ ; \quad y = 7t^3$$

$\frac{dz}{dx} = \cos(x^2+y^2) \cdot 2x$ $\qquad$ $\frac{dz}{dy} = \cos(x^2+y^2) \cdot 2y$

$\frac{dx}{dt} = 16t$ $\qquad$ $\frac{dy}{dt} = 21t^2$

Let's put it all together.

$$\frac{dz}{dt} = \frac{dz}{dx}\frac{dx}{dt} + \frac{dz}{dy}\frac{dy}{dt}$$
$$= 2x\cos(x^2+y^2) \cdot 16t + 2y\cos(x^2+y^2) \cdot 21t^2$$

And we are done b/c we want our answer in terms of $x, y,$ and $t$.

Example 2: Find $dz/dt$. Express your answer in terms of only $t$.

$$z = \frac{3x}{y} \quad ; \quad x = e^{-3t} \quad ; \quad y = 3t^2$$

Note $z = \frac{3x}{y} = 3xy^{-1}$

$$\frac{dz}{dx} = 3y^{-1} = \frac{3}{y} \qquad \frac{dz}{dy} = -3xy^{-2} = -\frac{3x}{y^2}$$

$$\frac{dx}{dt} = -3e^{-3t} \qquad \frac{dy}{dt} = 6t$$

Let's put it all together.

$$\frac{dz}{dt} = \frac{dz}{dx} \cdot \frac{dx}{dt} + \frac{dz}{dy} \cdot \frac{dy}{dt}$$

$$= \frac{3}{y} \cdot -3e^{-3t} + \left(-\frac{3x}{y^2}\right) 6t$$

$$= \frac{-9e^{-3t}}{y} - \frac{18xt}{y^2}$$

But this problem wants our answer in terms of ONLY $t$. So plug $x = e^{-3t}$ and $y = 3t^2$ into our formula

$$\frac{dz}{dt} = \frac{-9e^{-3t}}{3t^2} - \frac{18e^{-3t} \cdot t}{(3t^2)^2}$$

$$= \frac{-3e^{-3t}}{t^2} - \frac{18e^{-3t} \cdot t}{9t^4}$$

$$= \frac{-3e^{-3t}}{t^2} - \frac{2e^{-3t}}{t^3}$$

Example 3: Find dz/dt. Express your answer in terms of only t.

$z = x^2 y$ ; $x = \sin(7t)$ ; $y = \cos(3t)$

$\frac{dz}{dx} = 2xy$ $\qquad$ $\frac{dz}{dy} = x^2$

$\frac{dx}{dt} = 7\cos(7t)$ $\qquad$ $\frac{dy}{dt} = -3\sin(3t)$

Let's put it all together.

$$\frac{dz}{dt} = \frac{dz}{dx}\frac{dx}{dt} + \frac{dz}{dy}\frac{dy}{dt}$$
$$= 2xy\,(7\cos(7t)) + x^2(-3\sin(3t))$$
$$= 14xy\cos(7t) - 3x^2\sin(3t)$$

But this problem wants our answer in terms of ONLY t.
So plug $x = \sin(7t)$ and $y = \cos(3t)$ into our formula.

$$\frac{dz}{dt} = 14\sin(7t)\cos(3t)\cos(7t) - 3(\sin(7t))^2\sin(3t)$$

Example 4: Evaluate $\frac{dz}{dt}$ at $t=1$ when $z = \ln(3x^2+y)$, $x = t^{1/3}$ and $y = t^{2/3}$

$\frac{dz}{dx} = \frac{6x}{3x^2+y}$ $\qquad$ $\frac{dz}{dy} = \frac{1}{3x^2+y}$

$\frac{dx}{dt} = \frac{1}{3}t^{-2/3}$ $\qquad$ $\frac{dy}{dt} = \frac{2}{3}t^{-1/3}$

Before plugging all the equations into our formula, let's plug in $t=1$.

$x(1) = 1^{1/3} = 1$ $\qquad$ $\left.\frac{dx}{dt}\right|_{t=1} = \frac{1}{3}$ $\qquad$ $\left.\frac{dz}{dx}\right|_{(1,1)} = \frac{6}{3+1} = \frac{6}{4} = \frac{3}{2}$

$y(1) = 1^{2/3} = 1$ $\qquad$ $\left.\frac{dy}{dt}\right|_{t=1} = \frac{2}{3}$ $\qquad$ $\left.\frac{dz}{dy}\right|_{(1,1)} = \frac{1}{3+1} = \frac{1}{4}$

Let's put this all together

$$\left.\frac{dz}{dt}\right|_{t=1} = \left.\frac{dz}{dx}\right|_{(1,1)} \left.\frac{dx}{dt}\right|_{t=1} + \left.\frac{dz}{dy}\right|_{(1,1)} \left.\frac{dy}{dt}\right|_{t=1}$$

$$= \frac{1}{3} \cdot \frac{3}{2} + \frac{2}{3} \cdot \frac{1}{4}$$

$$= \frac{2}{3}$$

Example 5: Find $\left.\frac{dz}{dt}\right|_{t=1/4}$ if $z = x^2 - x\tan(y)$, $x = \sqrt{t}$, and $y = \pi t$

$$\frac{dz}{dx} = 2x - \tan(y) \qquad \frac{dz}{dy} = -x\sec^2(y)$$

$$\frac{dx}{dt} = \frac{1}{2}t^{-1/2} = \frac{1}{2\sqrt{t}} \qquad \frac{dy}{dt} = \pi$$

Before plugging all the equations into our formula, let's plug in $t = 1/4$.

$$x(1/4) = \sqrt{1/4} = 1/2 \qquad y(1/4) = \pi/4$$

$$\left.\frac{dx}{dt}\right|_{t=1/4} = \frac{1}{2\sqrt{1/4}} = \frac{1}{2(1/2)} = 1 \qquad \left.\frac{dy}{dt}\right|_{t=1/4} = \pi$$

$$\left.\frac{dz}{dx}\right|_{(1/2,\pi/4)} = 2\left(\frac{1}{2}\right) - \tan\left(\frac{\pi}{4}\right) = 1 - 1 = 0 \qquad \left.\frac{dz}{dy}\right|_{(1/2,\pi/4)} = -\frac{1}{2}\sec^2\left(\frac{\pi}{4}\right) = -1$$

Let's put this all together.

$$\left.\frac{dz}{dt}\right|_{t=1/4} = \left.\frac{dz}{dx}\right|_{(1/2,\pi/4)} \left.\frac{dx}{dt}\right|_{t=1/4} + \left.\frac{dz}{dy}\right|_{(1/2,\pi/4)} \left.\frac{dy}{dt}\right|_{t=1/4}$$

$$= 0 \cdot 1 + (-1)\pi$$

$$= -\pi$$