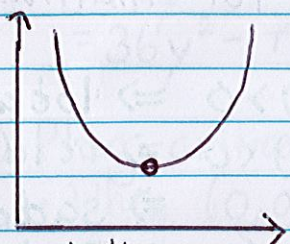
Lesson 23: Extrema of Functions of Two Variables

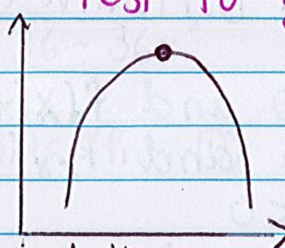
Recall from Calculus 1, when $y = f(x)$

- x_0 is a critical number if $f'(x_0) = 0$
- By the Second Derivative Test,
  - ↳ if $f''(x_0) > 0 \Rightarrow$ relative min @ $(x_0, f(x_0))$
  - ↳ if $f''(x_0) < 0 \Rightarrow$ relative max @ $(x_0, f(x_0))$
  - ↳ if $f''(x_0) = 0 \Rightarrow$ test is inconclusive

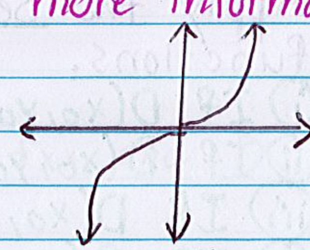
Typically we apply the First Derivative Test to gain more information



relative min



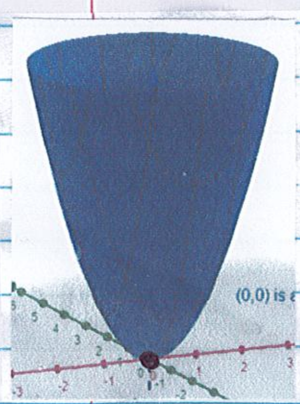
relative min



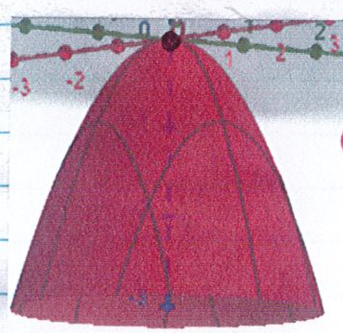
neither

Extrema for two variable function

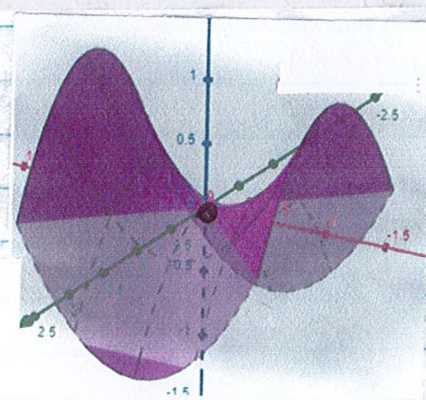
If $z = f(x,y)$, the following extrema may occur:



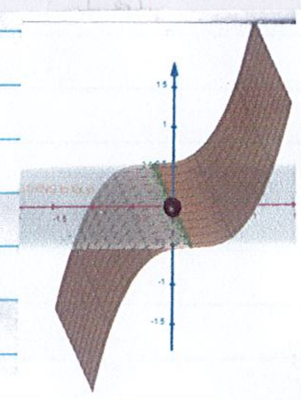
relative min



relative max



saddle point



neither

We see that in all the cases of extrema, the tangent plane to the graph is parallel to the xy-plane. We can describe this in terms of first order partials.

i.e. $\dfrac{df}{dx}(x_0, y_0) = 0$ or $\dfrac{df}{dy}(x_0, y_0) = 0$

To determine what type of extrema is taking place, we use can analogous Second Derivative Test for Multivariable Functions.

Finding extrema of functions of two variables

Let $z = f(x,y)$

① Find all the critical points.
i.e. All (x_0, y_0) such that $f_x(x_0,y_0) = 0$ and $f_y(x_0,y_0) = 0$

② Compute f_{xx}, f_{xy}, f_{yy} and
$$D = f_{xx} f_{yy} - (f_{xy})^2$$
where D is known as the discriminant.

③ For every given critical point (x_0,y_0), evaluate D and f_{xx} at (x_0,y_0)

④ Apply the Second Derivative Test for multivariable functions.

(i) If $D(x_0,y_0) > 0$ and $f_{xx}(x_0,y_0) > 0$ ⇒ relative min
(ii) If $D(x_0,y_0) > 0$ and $f_{xx}(x_0,y_0) < 0$ ⇒ relative max
(iii) If $D(x_0,y_0) < 0$ ⇒ saddle point
(iv) If $D(x_0,y_0) = 0$ ⇒ Test is inconclusive

Example 1: Find all the relative maxs, relative mins, and saddle points of the following functions:

ⓐ $f(x,y) = 3x^2y + y^3 - 3x^2 - 3y^2 + 2$

① Critical Point(s)
$$\begin{cases} f_x = 6xy - 6x = 0 & \text{I} \\ f_y = 3x^2 + 3y^2 - 6y = 0 & \text{II} \end{cases}$$

Solve I
$$6xy - 6x = 0$$
$$6x(y-1) = 0$$
$$x = 0 \quad y = 1$$

Plug $x=0$ and $y=1$ into II

• When $x = 0$,
$$3(0)^2 + 3y^2 - 6y = 0$$
$$3y^2 - 6y = 0$$
$$3y(y-2) = 0$$
$$y = 0,\ y = 2$$
Critical Pts: $(0,0)$, $(0,2)$

• When $y = 1$,
$$3x^2 + 3(1)^2 - 6(1) = 0$$
$$3x^2 + 3 - 6 = 0$$
$$3x^2 - 3 = 0$$
$$3(x^2 - 1) = 0$$
$$3(x-1)(x+1) = 0$$
$$x = \pm 1$$
Critical Pts: $(1,1)$, $(-1,1)$

② Compute f_{xx}, f_{xy}, f_{yy}, and D.

$f_x = 6xy - 6x$

$f_{xx} = 6y - 6$ $\qquad f_{xy} = 6x$

$f_y = 3x^2 + 3y^2 - 6y$

$f_{yy} = 6y - 6$

$$
\begin{aligned}
D &= f_{xx} f_{yy} - (f_{xy})^2 \\
&= (6y-6)(6y-6) - (6x)^2 \\
&= (6y-6)^2 - (6x)^2 \\
&= 36y^2 - 72y + 36 - 36x^2
\end{aligned}
$$

③ Plug critical points (from ①) into f_{xx} and D.

- When $(0,0)$,

  $f_{xx}(0,0) = 6(0) - 6 = -6$

  $D(0,0) = 36(0)^2 - 72(0) + 36 - 36(0)^2 = 36$

- When $(0,2)$,

  $f_{xx}(0,2) = 6(2) - 6 = 6$

  $D(0,2) = 36(2)^2 - 72(2) + 36 - 36(0)^2 = 36$

- When $(1,1)$,

  $f_{xx}(1,1) = 6(1) - 6 = 0$

  $D(1,1) = 36(1)^2 - 72(1) + 36 - 36(1)^2 = -36$

- When $(-1,1)$,

  $f_{xx}(-1,1) = 6(1) - 6 = 0$

  $D(-1,1) = 36(1)^2 - 72(1) + 36 - 36(-1)^2 = -36$

④ Second Derivative Test.

Using ③, we make the following conclusions:

- $(0,0)$: $D(0,0) = 36 > 0$ $\quad f_{xx}(0,0) = -6 < 0$ $\Rightarrow$ relative max
- $(0,2)$: $D(0,2) = 36 > 0$ $\quad f_{xx}(0,2) = 6 > 0$ $\Rightarrow$ relative min
- $(1,1)$: $D(1,1) = -36 < 0$ $\Rightarrow$ saddle pt
- $(-1,1)$: $D(-1,1) = -36 < 0$ $\Rightarrow$ saddle pt

ⓑ $f(x,y) = \frac{2}{3}y^3 + x^2 - 4yx - 10y + 6$

① Critical Point(s)

$f_x = 2x - 4y = 0$ Ⓘ

$f_y = 2y^2 - 4x - 10 = 0$ Ⓘ Ⓘ

Solve Ⓘ for x.

$2x - 4y = 0$

$2x = 4y$

$x = 2y$

Plug $x = 2y$ into Ⓘ Ⓘ

$2y^2 - 4(2y) - 10 = 0$

$2y^2 - 8y - 10 = 0$

$2(y^2 - 4y - 5) = 0$

$2(y-5)(y+1) = 0$

$y = 5, -1$

Plug $y = 5, -1$ into $x = 2y$

- $\underline{y=5}$

  $x = 2(5) = 10$

- $\underline{y=-1}$

  $x = 2(-1) = -2$

Hence the critical points are $(10,5), (-2,-1)$

② Compute f_{xx}, f_{xy}, f_{yy} and D.

$f_x = 2x - 4y$

$f_{xx} = 2$ $\qquad$ $f_{xy} = -4$

$f_y = 2y^2 - 4x - 10$

$f_{yy} = 4y$

$D = f_{xx}f_{yy} - (f_{xy})^2$

$= 2(4y) - (-4)^2$

$= 8y - 16$

③ Plug critical points (from ①) into f_{xx} and D.

- $\underline{(10,5)}$: $f_{xx}(10,5) = 2$

  $D(10,5) = 8(5) - 16 = 34$

- $\underline{(-2,-1)}$: $f_{xx}(-2,-1) = 2$

  $D(-2,-1) = 8(-1) - 16 = -24$

④ Second Derivative Test

Using ③, we make the following conclusions:

- $\underline{(10,5)}$: $D(10,5) = 34 > 0$ $\quad f_{xx}(10,5) = 2 > 0$ $\Rightarrow$ relative min
- $\underline{(-2,-1)}$: $D(-2,-1) = -24 < 0$ $\Rightarrow$ saddle point

ⓒ $f(x,y) = \frac{3}{2}x^4 - yx^2 + 20x^2 + \frac{1}{2}y^2 - 3$

① Critical Point(s)

$f_x = 6x^3 - 2xy + 40x$ Ⓘ

$f_y = -x^2 + y = 0$ Ⓘⓘ

Solve Ⓘⓘ for y.

$-x^2 + y = 0$

$y = x^2$

Plug $y = x^2$ into Ⓘ.

$6x^3 - 2x(x^2) + 40x = 0$

$6x^3 - 2x^3 + 40x = 0$

$4x^3 + 40x = 0$

$4x(x^2+10) = 0$

$4x = 0 \quad \underbrace{x^2+10=0}$

$x = 0 \quad$ no solution

Plug $x=0$ into $y = x^2$

$y = 0^2 = 0$

Hence $(0,0)$ is the only critical point.

② Compute f_{xx}, f_{xy}, f_{yy}, and D.

$f_x = 6x^3 - 2xy + 40x$

$f_{xx} = 18x^2 - 2y + 40 \qquad f_{xy} = -2x$

$f_y = -x^2 + y$

$f_{yy} = 1$

$D = f_{xx}f_{yy} - (f_{xy})^2$

$= (18x^2 - 2y + 40)(1) - (-2x)^2$

$= 18x^2 - 2y + 40 - 4x^2$

$= 14x^2 - 2y + 40$

③ Plug critical point (from ①) into f_{xx} and D.

(0,0): $f_{xx}(0,0) = 18(0)^2 - 2(0) + 40 = 40$

$D(0,0) = 14(0)^2 - 2(0) + 40 = 40$

④ Second Derivative Test

Using ③, we make the following conclusion:

(0,0): $D(0,0) = 40 > 0$ $f_{xx}(0,0) = 40 > 0$ $\Rightarrow$ relative min