MA 16020 LESSON 26: LAGRANGE MULTIPLIERS CONSTRAINED MIN/MAX (HANDOUT)

How do we solve Optimization Problems via LaGrange Multipliers?

- Determine an **objective function,** $f(x, y)$, that we need to maximize or minimize.

- Determine a **constraint equation,** $g(x, y) = k$.

- Then we can solve for the maximum or minimum via **the Method of LaGrange Multiplers** from Last Class.
    1. Solve the following system of equations.
$$f_x = \lambda g_x$$
$$f_y = \lambda g_y$$
$$g(x, y) = k$$

    2. Evaluate f at every point (x, y) found in (1).
        a. The largest of these is the maximum value of f.
        b. The smallest of these is the minimum value of f.

    Note the equations from (1) are known as **LaGrange Equations** and the constant, λ, is called the **LaGrange Multiplier.**

- Reread the question and be sure you have answered exactly what was asked.

Example 1: On a certain island, at any given time, there are R hundred rats and S hundred snakes. Their population are related by the equation:

$$(R-18)^2 + (S-14)^2 = 72$$

What is the maximum combined number of snakes and rats that could ever be on this island at the same time?

Let $f(R,S) = R+S$ and $g(R,S) = (R-18)^2 + (S-14)^2 = 72$

Using LaGrange Multipliers,

$f_R = 1 \qquad \lambda g_R = \lambda(2(R-18))$

$f_S = 1 \qquad \lambda g_S = \lambda(2(S-14))$

Our system is

$$\begin{cases} 1 = 2(R-18)\lambda & ① \\ 1 = 2(S-14)\lambda & ② \\ (R-18)^2 + (S-14)^2 = 72 & ③ \end{cases}$$

Multiply ① by $(S-14)$ and ② by $(R-18)$.

$$\begin{cases} S-14 = 2(R-18)(S-14)\lambda & ① \\ R-18 = 2(R-18)(S-14)\lambda & ② \end{cases}$$

Set ① and ② equal

$$S-14 = R-18$$
$$S+4 = R$$

Plug $R = S+4$ into ③.

$$(S+4-18)^2 + (S-14)^2 = 72$$
$$(S-14)^2 + (S-14)^2 = 72$$
$$2(S-14)^2 = 72$$
$$(S-14)^2 = 36$$
$$S-14 = \pm 6$$
$$S = 14 \pm 6$$

So $S = 20$ or $S = 8$

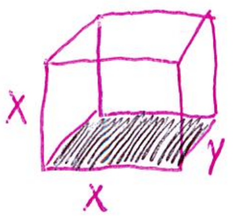
When $S = 20$, $R = 24$
$S = 8$, $R = 12$

Test for Max

- $f(24,20) = 44 \Rightarrow \text{max}$
- $f(12,8) = 20$

We want the maximum combined number of snakes and rats. So $f(24,20) = 44$

Example 2: **A rectangular building with a square front is to be constructed of materials that costs \$18 per ft^2 for the flat root, \$16 dollars per ft^2 for the sides and the back, and \$ 9 per ft^2 for the glass front. We will ignore the bottom of the building. If the volume of the building is 64000ft^3, what dimensions will minimize the cost of materials?**



$$\begin{cases} f(x,y) = 18xy + 16(x^2 + 2xy) + 9x^2 = 50xy + 25x^2 \\ g(x,y) = x^2y = 64000 \end{cases}$$

Using LaGrange Multipliers,

$f_x = 50y + 50x \qquad \lambda g_x = \lambda(2x) = 2xy\lambda$

$f_y = 50x \qquad \lambda g_y = \lambda(x^2) = x^2\lambda$

Our System is

$$\begin{cases} 50y + 50x = 2xy\lambda \\ 50x = x^2\lambda \\ x^2y = 64000 \end{cases} \xleftrightarrow{b/c\ x \neq 0} \begin{matrix} 25y + 25x = xy\lambda & ① \\ 50 = x\lambda & ② \\ x^2y = 64000 & ③ \end{matrix}$$

Multiply ② by y.

$50y = xy\lambda$ ②

Set ① and ② equal

$25y + 25x = 50y$

$25x = 25y$

$x = y$

Plug $x = y$ into ③.

$x^3 = 64000$

$x = 40$

Since $x = y$, $y = 40$.

We want the dimensions. So

$l \times w \times h = 40 \times 40 \times 40$.

Example 3: **Alice will use all of the last 24 hours before her exam to study two different ways. Without any preparation she would earn 300 points out of 1000 points on the exam. It is estimated that her exam score will improve by $x(59 - x)$ points if she reads her lecture notes for x hours and $y(51 - y)$ points if she solves review problems for y hours, but due to fatigue she will lose $(x + y)^2$ points. What is the maximum exam score she can obtain?**

Let $f(x,y) = x(59-x) + y(51-y) - (x+y^2) + 300$

$= 59x - x^2 + 51y - y^2 - (x+y)^2 + 300$

and $g(x,y) = x + y = 24$.

Using LaGrange Multipliers,

$f_x = 59 - 2x - 2(x+y)$ $\qquad \lambda g_x = \lambda(1) = \lambda$

$f_y = 51 - 2y - 2(x+y)$ $\qquad \lambda g_y = \lambda(1) = \lambda$

Our system is

$$\begin{cases} 59 - 2x - 2(x+y) = \lambda \\ 51 - 2y - 2(x+y) = \lambda \\ x + y = 24 \end{cases} \Rightarrow \begin{cases} 59 - 2x = \lambda + 2(x+y) & ① \\ 51 - 2y = \lambda + 2(x+y) & ② \\ x + y = 24 & ③ \end{cases}$$

Set ① and ② equal.

$59 - 2x = 51 - 2y$

$8 - 2x = -2y$

$2x - 8 = 2y$

$x - 4 = y$

Plug $y = x - 4$ into ③.

$x + y = 24$

$x + x - 4 = 24$

$2x = 28$

$x = 14$

Plug $x = 14$ into

$y = x - 4$

$y = 14 - 4$

$y = 10$

We want the maximum exam score

$f(14, 10) = 764$