Lesson 27: Double Integrals

Let $z = f(x,y)$ be a function of two variables. Similar to taking partial derivatives with respect to x and y, we can take

- $\int_{x=a}^{x=b} f(x,y)\,dx$ — Integrate with respect to x and Treat y as a constant
- $\int_{y=c}^{y=d} f(x,y)\,dy$ — Integrate with respect to y and Treat x as a constant

Combining the above integrals, we obtain the following double integrals:

- $\int_{y=c}^{y=d}\int_{x=a}^{x=b} f(x,y)\,dx\,dy = \int_{y=c}^{y=d}\left(\int_{x=a}^{x=b} f(x,y)\,dx\right)dy$
- $\int_{x=a}^{x=b}\int_{y=c}^{y=d} f(x,y)\,dy\,dx = \int_{x=a}^{x=b}\left(\int_{y=c}^{y=d} f(x,y)\,dy\right)dx$

The stragedy to evaluate these integrals is to first integrate the inner integral and work outwards.

Example 1: Evaluate

ⓐ $\int_1^2\int_0^1 x^2y\,dy\,dx = \int_{x=1}^{x=2}\left(\int_{y=0}^{y=1} x^2y\,dy\right)dx$

$= \int_{x=1}^{x=2}\left(x^2\cdot\frac{y^2}{2}\Big]_{y=0}^{y=1}\right)dx$

$= \int_{x=1}^{x=2}\left(x^2\left(\frac{1^2}{2}-\frac{0^2}{2}\right)\right)dx$

$= \int_{x=1}^{x=2}\left(x^2\cdot\frac{1}{2}\right)dx$

$= \int_{x=1}^{x=2}\frac{x^2}{2}\,dx$

$= \frac{1}{2}\cdot\frac{x^3}{3}\Big]_{x=1}^{x=2}$

$= \frac{1}{6}(2^3-1^3) = \frac{7}{6}$

(b) $$\int_5^6 \int_0^y 8xy\,dx\,dy = \int_{y=5}^{y=6}\left(\int_{x=0}^{x=y} 8xy\,dx\right)dy$$

$$= \int_{y=5}^{y=6}\left(8y \cdot \frac{x^2}{2}\Big]_{x=0}^{x=y}\right)dy$$

$$= \int_{y=5}^{y=6}\left(4y \cdot x^2\Big]_{x=0}^{x=y}\right)dy$$

$$= \int_{y=5}^{y=6}\left(4y\left(y^2-0^2\right)\right)dy$$

$$= \int_{y=5}^{y=6} 4y^3\,dy$$

$$= \frac{4y^4}{4}\Big]_{y=5}^{y=6}$$

$$= y^4\Big]_{y=5}^{y=6}$$

$$= 6^4 - 5^4 = 671$$

(c) $$\int_{4\pi}^{8\pi}\int_0^y -11\csc(y)\cos(x)\,dx\,dy$$

$$= \int_{y=4\pi}^{y=8\pi}\left(\int_{x=0}^{x=y} -11\csc(y)\cos(x)\,dx\right)dy$$

$$= \int_{y=4\pi}^{y=8\pi}\left(-11\csc(y)\cdot(-\sin(x))\Big]_{x=0}^{x=y}\right)dy$$

$$= \int_{y=4\pi}^{y=8\pi}\left(11\csc(y)\cdot\sin(x)\Big]_{x=0}^{x=y}\right)dy$$

$$= \int_{y=4\pi}^{y=8\pi}\left(11\csc(y)\left(\sin(y) - \underbrace{\sin(0)}_{=0}\right)\right)dy$$

$$= \int_{y=4\pi}^{y=8\pi}\left(11\csc(y)\sin(y)\right)dy$$

Remember $\csc(y)\sin(y) = 1$.

$$= \int_{y=4\pi}^{y=8\pi} 11\,dy$$

$$= 11y\Big]_{y=4\pi}^{y=8\pi}$$

$$= 11(8\pi - 4\pi) = 11\cdot 4\pi = 44\pi$$

(d) $$\int_1^e \int_0^{9\ln x} 5x\,dy\,dx = \int_{x=1}^{x=e} \left(\int_{y=0}^{y=9\ln x} 5x \cdot 1\,dy \right) dx$$

$$= \int_{x=1}^{x=e} \left(5x \cdot y \Big]_{y=0}^{y=9\ln x} \right) dx$$

$$= \int_{x=1}^{x=e} \left(5x \cdot \left(9\ln x - 0 \right) \right) dx$$

$$= \int_{x=1}^{x=e} 45x \ln x\,dx$$

$$\frac{u = \ln x \qquad dv = x\,dx}{du = {}^{1}/_{x}\,dx \qquad v = x^2/2} \quad 45\left(\frac{x^2 \ln x}{2} \Big]_{x=1}^{x=e} - \int_{x=1}^{x=e} \frac{x^2}{2} \cdot \frac{1}{x}\,dx \right)$$

$$= 45\left(\frac{x^2 \ln x}{2} \Big]_{x=1}^{x=e} - \int_{x=1}^{x=e} \frac{x}{2}\,dx \right)$$

$$= 45\left(\frac{x^2 \ln x}{2} \Big]_{x=1}^{x=e} - \frac{1}{2} \cdot \frac{x^2}{2} \Big]_{x=1}^{x=e} \right)$$

$$= 45\left(\frac{e^2 \ln(e)}{2} - \underbrace{\frac{1^2 \ln(1)}{2}}_{=0} - \frac{1}{4}\left(e^2 - 1^2\right) \right)$$

$$= 45\left(\frac{e^2}{2} - \frac{e^2}{4} + \frac{1}{4} \right)$$

$$= 45\left(\frac{e^2}{4} + \frac{1}{4} \right)$$