Lesson 28: Double Integrals

Last Class, we introduced

$$\int_{y=c}^{y=d}\left(\int_{x=a}^{x=b} f(x,y)\,dx\right)dy \text{ and } \int_{x=a}^{x=b}\left(\int_{y=c}^{y=d} f(x,y)\,dy\right)dx$$

A geometric interpertation of these double integrals is we are finding the volume below $z=f(x,y)$ above the region

$$R=\{(x,y) \mid a\le x\le b;\ c\le y\le d\}$$

We can denote the integrals above by just one

$$\iint_R f(x,y)\,dA$$

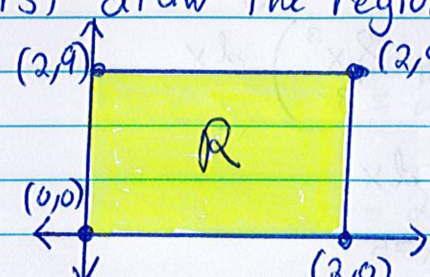
where R is the domain of integration.

To solve the integrals of the form $\iint_R f(x,y)\,dA$, we start by drawing the region. We do this to determine the bounds of our integrals.

Example 1: Evaluate the integral $\iint_R 10x^3y\,dA$ where R is the rectangle with vertices $(0,0)$, $(2,0)$, $(0,9)$, and $(2,9)$.

First draw the region, R. We can see that $0\le x\le 2$ and $0\le y\le 9$. So

$$R=\{(x,y) \mid 0\le x\le 2,\ 0\le y\le 9\}$$



Hence $$\iint_R 10x^3y\,dA=\int_{x=0}^{x=2}\left(\int_{y=0}^{y=9} 10x^3\cdot y\,dy\right)dx$$

$$=\int_{x=0}^{x=2}\left(10x^3\cdot \frac{y^2}{2}\Big]_{y=0}^{y=9}\right)dx$$

$$=\int_{x=0}^{x=2}\left(10x^3\left(\frac{9^2}{2}-\frac{0^2}{2}\right)\right)dx$$

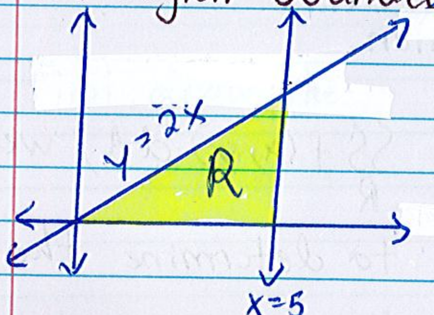
$$= \int_{x=0}^{x=2} \left(10x^3 \cdot \frac{81}{2}\right) dx$$

$$= \int_{x=0}^{x=2} 405x^3\, dx$$

$$= \frac{405}{4} x^4 \Big]_{x=0}^{x=2}$$

$$= \frac{405}{4}(2^4 - 0^4) = 1620$$

Example 2: Evaluate the integral $\iint_R (x^2+y^2)\, dA$ where R is the region bounded by the lines $y=2x$, $x=5$, and the x-axis.



First draw the region, R. We can see that

$$R = \{(x,y) \mid 0 \le x \le 5,\ 0 \le y \le 2x\}$$

Hence $\iint_R (x^2+y^2)\, dA = \int_{x=0}^{x=5} \left(\int_{y=0}^{y=2x} (x^2+y^2)\, dy \right) dx$

$$= \int_{x=0}^{x=5} \left(\left(x^2 \cdot y + \frac{y^3}{3}\right)\Big]_{y=0}^{y=2x} \right) dx$$

$$= \int_{x=0}^{x=5} \left(x^2(2x) + \frac{(2x)^3}{3} - \left(x^2 \cdot 0 + \frac{0^3}{3}\right)\right) dx$$

$$= \int_{x=0}^{x=5} \left(2x^3 + \frac{8x^3}{3}\right) dx$$

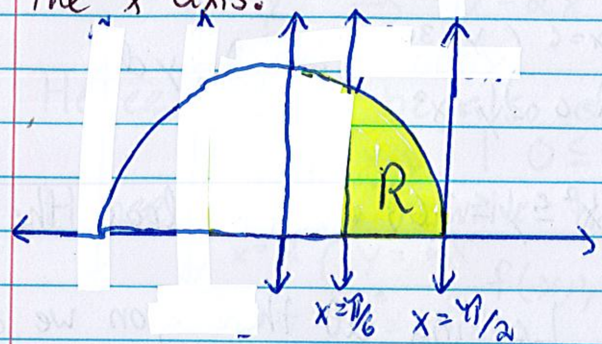
$$= \int_{x=0}^{x=5} \frac{14}{3} x^3\, dx$$

$$= \frac{14}{3} \cdot \frac{x^4}{4} \Big]_{x=0}^{x=5}$$

$$= \frac{7}{6} x^4 \Big]_{x=0}^{x=5}$$

$$= \frac{7}{6}(5^4 - 0^4) = \frac{4375}{6}$$

Example 3: Evaluate the integral $\iint_R 6\sin^2(x)\,dA$ where R is the region bounded by the curves $y=\cos(x)$, $x=\pi/6$, $x=\pi/2$ and the x-axis.



First draw the region, R. We can see that

$$R=\left\{(x,y)\,\middle|\,\frac{\pi}{6}\le x\le\frac{\pi}{2},\ 0\le y\le\cos x\right\}$$

Hence
$$\iint_R 6\sin^2(x)\,dA=\int_{x=\pi/6}^{x=\pi/2}\left(\int_{y=0}^{y=\cos(x)}6(\sin x)^2\,dy\right)dx$$
$$=\int_{x=\pi/6}^{x=\pi/2}\left(6(\sin x)^2\cdot y\Big]_{y=0}^{y=\cos x}\right)dx$$
$$=\int_{x=\pi/6}^{x=\pi/2}\left(6(\sin x)^2\cdot(\cos x-0)\right)dx$$
$$=\int_{x=\pi/6}^{x=\pi/2}6(\sin x)^2\cdot\cos x\,dx$$

$$\frac{u=\sin x}{du=\cos x\,dx}\quad\int 6u^2\,du$$
$$=\frac{6u^3}{3}=2u^3$$
$$=2(\sin x)^3\Big]_{x=\pi/6}^{x=\pi/2}$$
$$=2\left(\left(\sin\left(\frac{\pi}{2}\right)\right)^3-\left(\sin\left(\frac{\pi}{6}\right)\right)^3\right)$$
$$=2\left((1)^3-\left(\frac{1}{2}\right)^3\right)=2\left(1-\frac{1}{8}\right)=2\cdot\frac{7}{8}=\frac{7}{4}$$

Note that Examples 2 and 3 could have been done with dxdy as the order of integration.

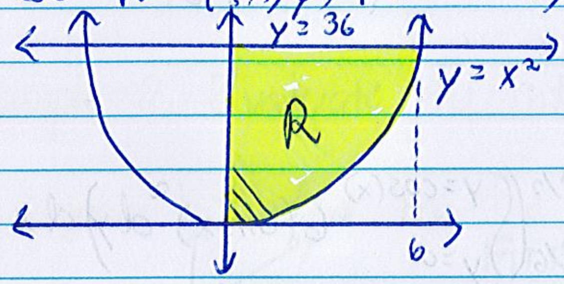
So a good question to ask is when to use dxdy or dydx? We will answer that next time. But in prepartion of that let's pratice switching the order of integration.

To switch the order of integration, we need to draw a picture.

Example 4: Switch the order of integration on the following integrals.

ⓐ $\int_0^6 \int_{x^2}^{36} f(x,y)\,dy\,dx = \int_{x=0}^{x=6} \int_{y=x^2}^{y=36} f(x,y)\,dy\,dx$

So $R = \{(x,y) \mid 0 \le x \le 6,\ x^2 \le y \le 36\}$. Let's draw the region.



Looking at the region we can describe R in another way. The y-values are $0 \le y \le 6$. As for x, we see that x lies between the y-axis $(x=0)$ and the parabola $(y=x^2)$. Let's get $y=x^2$ in terms of $x =$ ___.

$x = \pm\sqrt{y}$

Since $x \ge 0$, $x = \sqrt{y}$. Hence R can be also described by

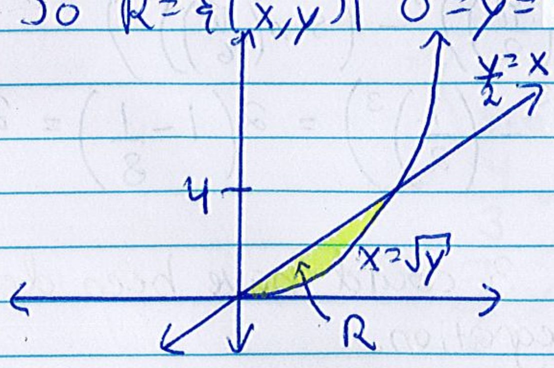
$R = \{(x,y) \mid 0 \le y \le 6,\ 0 \le x \le \sqrt{y}\}$

Hence we can rewrite the integral to be

$$\int_{y=0}^{y=6} \int_{x=0}^{x=\sqrt{y}} f(x,y)\,dx\,dy$$

ⓑ $\int_0^4 \int_{y/2}^{\sqrt{y}} f(x,y)\,dx\,dy = \int_{y=0}^{y=4} \int_{x=y/2}^{x=\sqrt{y}} f(x,y)\,dx\,dy$

So $R = \{(x,y) \mid 0 \le y \le 4,\ y/2 \le x \le \sqrt{y}\}$. Let's draw R.



Looking at the region we can describe R in another way. We see $x=0$ is the smallest value to find the largest plug $y=4$ into

$x = \frac{y}{2}$ or $x = \sqrt{y}$.

So $x = \frac{4}{2} = 2$. Hence $0 \le x \le 2$. As for y, we see that y

lies between $x=\sqrt{y}$ and $x=\frac{y}{2}$. Let's get both equations in terms of y.

$x=\frac{y}{2} \Rightarrow y=2x$ $\quad\quad$ $x=\sqrt{y} \Rightarrow y=x^2$

Hence R can be also described by

$R=\{(x,y) \mid 0 \leq x \leq 2,\ x^2 \leq y \leq 2x\}$

Hence we can rewrite the integral to be

$$\int_{x=0}^{x=2}\int_{y=x^2}^{y=2x} f(x,y)\,dy\,dx$$